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W. L. Brown.

ELEMENTS

OF

MECHANICAL PHILOSOPHY,

FOR THE USE OF THE

JUNIOR STUDENTS OF THE UNIVERSITY OF VIRGINIA.

BY

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MECHANICAL PHILOSOPHY.

CHAPTER I.

OF PHYSICAL SCIENCE IN GENERAL, AND THE CLASSIFICATION OF ITS PRINCIPAL SUBDIVISIONS.

1. Twofold division of Science into Mental and Physical. — The entire body of positive science comprises two great classes of truths. The first consists of those which are derived from consciousness, self-analysis, and the study of intellectual phenomena in general. These, when arranged according to their proper relations, constitute Mental Science; which includes the purely Logical Sciences, together with Intellectual Philosophy, or the Science of Mind in the abstract, and Moral Philosophy, or the Science of the Emotions as connected with human conduct. The second class of truths comprehends those which are deduced from the observation of material bodies, and the phenomena of the external world in the natural course of physical events, in experiments, and in the processes of the arts. These, reduced to their appropriate order, constitute Physical Science.

2. The following Table embraces all the great subdivisions of Physical Science. This, with the accompanying explanation of the scope and object of each department, will, it is thought, convey to the student a just conception of their relations to one another in a classified system; and will thus, as it were, indicate to him the position and extent, on the great map of knowledge, of the particular province in which his inquiries are to be conducted.

Tabular Classification of Physical Science.

A. NATURAL HISTORY.	B. NATURAL PHILOSOPHY.
1. Zoology,	1. Mechanical Philosophy,
2. Botany,	2. Chemistry,
3. Mineralogy,	3. Geology,
4. Anatomy, Vegetable and animal.	4. Physiology, Vegetable and animal.

3. Material bodies may be investigated either as permanent or undergoing a change, as quiescent or in motion. Every change is unavoidably regarded as an effect, and is referred to some cause which in physics is called a Force. These are all the relations in which material objects can be contemplated or studied. Hence the whole of physical science is embraced in two great subdivisions — *First*, the relations of objects *as they exist in space*, such as their shape, dimensions, weight, color and structure, and their resemblances and differences in these respects; and *Secondly*, their relations as *cause and effect*, or their actions, the one upon the other, producing motion, and various other changes.

The former of these subdivisions includes the truths which fall especially within the province of Natural History; the latter, those which in an enlarged use of the term may be embraced under the head of Natural Philosophy.

4. Thus understood, Natural History is a simple description of natural objects in their relations of form, color, structure, and other features, with an arrangement or classification of them according to these relations. Natural philosophy, in the extended sense here implied, is an account of the actions of bodies upon one another, or of the changes they mutually produce; in other words, of the causes and effects in the material world.

5. In adopting this twofold subdivision of physical science, it should be remembered, that as the two classes of relations marking respectively these subdivisions are inseparably blended in nature, it is impossible to treat of one without in some degree referring to the other. Thus the natural history of minerals unavoidably includes an account of the

mechanical and chemical actions or properties of each substance described, — such, for example, as its hardness and elasticity, its changes when exposed to heat or liquid solvents, or the color it imparts to flame. So, conversely, in a systematic treatise upon the various chemical reactions amongst bodies, which forms a part of Natural Philosophy, the simpler relations of each, as an object of Natural History, forms an indispensable part of the description. In a word, every system of mineralogy includes many chemical details, and every system of chemistry many particulars proper to mineralogy.

6. So again, in the natural history of animals and plants, a large space is allotted to the description of physiological actions, and other phenomena of change which appertain to the other subdivision, but which are, nevertheless, requisite to a complete knowledge and proper classification of these objects.

7. But, even in the blended form in which the two great divisions of physics are always and necessarily investigated, their respective objects are recognized as distinct. The leading aim of Natural History is the *classification of objects*, and it includes the study of actions and changes only as subordinate to this end. The great purpose of Natural Philosophy is the discovery of the *laws according to which the various changes of the material world are produced.* and it classifies actions only as a step to this result.

8. The several divisions of Natural History are named with reference to the nature of the objects they include. Thus, ZOOLOGY is the natural history of animals; BOTANY, that of plants; MINERALOGY, that of mineral bodies; and ANATOMY is a classified description of the parts or organs of animals and plants.

9. In order to apprehend clearly the distinction between MECHANICAL PHILOSOPHY AND CHEMISTRY, it is necessary to observe, that while they both treat of the actions or changes among bodies, and have in view to discover the laws according to which they are produced, the changes which they consider are not of the same kind. *All physical actions may be arranged in two classes,* — those which are not attended by a change in the properties of the bodies con-

cerned, and those in which these properties are either modified, or wholly transformed. The former may be termed Mechanical Actions; the latter are Chemical Actions.

We have examples of *mechanical action* in the pulling force exerted by a stone suspended by a cord, in its descent to the ground, when the cord is cut, in the impulsion of a boat by oars, or of a ship by the wind, in the rebound of a foot-ball, in the vibration of a stretched string, and in the deflexion of a compass-needle by an approaching bar of iron. In all such cases, it is apparent that the bodies concerned in the action are not themselves changed by it.

Illustrations of *chemical action* are seen in the calcination of chalk, or limestone, by heat, whereby it is converted into caustic lime; in the solution of the same substance in muriatic acid, causing it to disappear, and form a bitter compound; in the formation of black ink, from a mixture of nearly colorless solutions of copperas and nut-galls; in the explosion of gunpowder, by a spark; in the formation of a solid from two liquids, or of a liquid from two solids. In these instances, the bodies originally brought together are so changed by their mutual action as to be no longer recognized. The parts of which they are composed have been thrown into new arrangements, and thus new bodies have been constituted, which, while they contain all the elements of the former, are endowed with entirely different properties.

The two kinds of action are seen in succession, in the discharge of a loaded cannon. First, the spark causes the gunpowder to inflame, and thus, by a new arrangement of its elements, to assume rapidly the condition of highly elastic vapor and air. This is a chemical action and change. But the suddenly generated vapor and air, by their violent expansion, press upon the ball, and urge it forward with increasing speed, until it escapes from the muzzle of the piece; and this is a mechanical action.

10. These considerations lead us at once to an appropriate definition of the objects of Mechanical Philosophy and Chemistry.

MECHANICAL PHILOSOPHY investigates all those actions and changes which are unaccompanied by an alteration in the bodies concerned.

CHEMISTRY investigates all those actions and changes which permanently modify or entirely transform the bodies.

11. It will be seen, from the preceding examples, that mechanical actions necessarily involve motion or the tendency to motion, and nothing more. The motion may either be shared equally by all parts of the moving mass, as when a heavy body falls, or it may be unequally distributed, as in the case of a vibrating string or a revolving wheel. But in no case of mechanical action does the displacement give rise to new arrangements or combinations among the parts of the bodies. On the other hand, in chemical actions, the motions are always attended with combinations and decompositions which originate new bodies.

12. GEOLOGY has for its object to investigate the composition and arrangement of the great mineral masses composing the earth's crust, the various chemical and mechanical actions by which they are or have been modified, and the peculiar features of topography and climate, as well as the different races of animals and plants, which, in adaptation to these, have marked the successive periods of the earth's history. In a word, this science proposes to describe all the great phases in which the mineral materials and organic inhabitants of the globe have successively presented themselves, and to investigate the physical agencies by which they have been produced or accompanied. In this wide range of subjects, we see that Geology embraces much that belongs to Natural History as well as Natural Philosophy. The descriptions and classifications of Zoology and Botany, no less than those of Mineralogy; and the laws of physical action, in the organic as well as inanimate world, which are embraced in Chemistry, Mechanics and Physiology, form the basis of geological investigation; and indeed Geology is but the application of these facts and principles to the great problem of the physical and organic history of the globe.

13. The science of PHYSIOLOGY proposes to investigate the phenomena of living structures, and the laws by which these phenomena are governed. It therefore studies the functions of their different organs in nutrition, growth, and other vital actions, as well as the sources and changes of the materials

out of which they are constructed; marks the development of particular parts of entire living beings and of races, with the influence of climate and other external agencies upon them; and labors to trace the actions and transformations of living nature to the chemical, mechanical, or other laws on which they depend.

The various organs of the living machine, and the structures composing them, are described and classified by Anatomy; but Physiology informs us of their origin, uses, and development.

CHAPTER II.

ON THE SOURCE AND FOUNDATION OF PHYSICAL SCIENCE.

OBSERVATION.—INDUCTION.—DEDUCTION.

14. **OBSERVATION NOT OF ITSELF SUFFICIENT TO CONSTITUTE SCIENCE.** — BY observation is here meant the study of physical phenomena, whether in the ordinary course of nature or in the processes of the arts, and it therefore includes all that is embraced in the term experiment as well as observation proper.

While in this enlarged sense observation is always an essential step to the attainment of physical truth, certain internal acts of thought are equally indispensable. Observation in its various modes of application furnishes the data or materials of physical science. These of themselves would be nothing more than isolated impressions in the mind, apparently destitute of relations to one another and to the great system of things. But the mind, by its own inherent activity, discovers their relations, connects fact with fact, and infers general facts or laws; compares object with object, and classifies all objects in appropriate order. It has been well said, that experience is the beginning and occasion of all knowledge; but it is not knowledge before it has been moulded into a new and better form by the innate powers of the mind itself.

15. Observation is partly Subjective and partly Objective. — Our observations, simply considered, are but sensations excited in us by external objects. But it is a law of thought, that every impression from without is accompanied by some feeling or idea of relation, which forms, in fact, the essential condition of our perception. Thus we cannot see one object without the idea of space or form, we cannot see two without the idea of resemblance or difference; and yet space and difference are not sensations. Thus, in our observation of nature, there is always some act of thought which depends on ourselves, and some matter of thought which is independent of us; in a word, every part of our knowledge of the external world is made up of a *subjective* and *objective* element.

16. Elementary ideas, which lie at the foundation of physical science.

The ideas of relation, which [illegible] fundamental to the physical sciences are the ideas [illegible], Number, Time, Cause and Effect, including Matter and Force, Resemblance, Relation of means to ends, and others which are appropriate to various special branches of science.

17. Of Deduction. — Deductive Sciences. — Pure Mathematics. — In the investigation of truth, two distinct modes of procedure are followed, each having its appropriate object and end. The first is the deductive method, in which truths are deduced by a logical process from the fundamental ideas and depositions of the science to which they belong. The second is the inductive method, according to which laws or general truths are attained through the observation and comparison of many particular phenomena.

The relations of Space and Number belong equally to all external objects and events. These simple and comprehensive ideas give rise by their development to a body of science, applicable to objects and changes of all kinds. In this are included Geometry, the Theory of Numbers, Algebra regarded as the science of symbols, and indeed the whole of the *Pure Mathematics*. These are called pure sciences, because they are unfolded out of their proper fundamental ideas, through the axioms and definitions framed in accordance with them. The process of thought, by which conclusions are thus evolved, is that of Deduction.

The demonstration of the truths belonging to these sciences, has no further dependence on experience or observation than what is implied in the necessary connection of the ideas of space and number with our early external impressions. Observation does not add to the force of such a demonstration, for the truth is felt to be a logical and necessary deduction from the axioms and definitions of the science.

18. Of Physico-Mathematical Science. — It follows from what has just been said, that the Pure Mathematics, although indebted for its fundamental ideas to a simple and universal experience, ought rather to be classed with the logical sciences than with the sciences of observation. But the consideration of mathematical relations enters largely into many of the sciences which treat of the phenomena and laws of the material world; and this, it is obvious, must be the case, wherever [illegible] investigation has reference to precise measures of the distance, magnitude, figure, direction, or motion of bodies. Hence, practically considered, we cannot separate mathematical from physical truth. In certain of the physical sciences the phenomena are described in mathematical language, and their most comprehensive truths are deduced from the observed facts by a consideration of the mathematical relations which belong to them. These sciences are hence called Physico-Mathematical sciences.

19. Nature of Induction and of the Inductive Sciences.— It has already been stated that all physical science is founded on observation. But it is also true that observations alone do not make science. It is by applying to observations the inductive process of thought that we discover the laws of nature, the greater and lesser generalizations of the physical sciences. In this procedure, as already mentioned, the facts collected by observation are sifted, analyzed, compared and grouped together, according to their resemblances, or their relation as cause and effect; and from this comparison and grouping, the mind at once rises to the classifications

of Natural History, and to the laws of physical action which belong to Natural Philosophy.

20. The following examples may serve to illustrate the inductive method of inquiry:

(*a*). It is found by experiment that the two ingredients of water, oxygen and hydrogen, are always present in that body, in the proportion of eight parts, by weight, of the former, to one of the latter. Experiment proves also that a like constancy obtains in the proportion of the ingredients of other chemical compounds — a special numerical ratio belonging to each. From a multitude of such observations, we gather the great inductive fact or law, that in bodies whose ingredients are chemically united, these ingredients are always present in certain fixed proportions.

(*b*). When a ray of light, moving through empty space, falls obliquely upon a horizontal surface of water or clear glass, and is transmitted, we find that it does not continue in its former direction as it moves through the new medium, but deviates into a line which lies nearer to the perpendicular drawn through the point of incidence. We observe a similar result, when, for the water or glass, we substitute any other transparent solid or liquid. Induction converts our observations into the general fact or law, that transparent bodies bend or refract the rays of light, impinging upon them so as to cause them to take a direction more nearly perpendicular to the surface. But a closer observation shows that in each medium, whatever be the obliquity of the incident ray, its direction and that of the refracted ray are connected by a fixed numerical relation, and, moreover, that this numerical relation is different for different media. We have thus a second and more precise induction, which forms one of the fundamental laws of Optics, viz., that the sine of the angle of incidence in each medium bears a fixed ratio to the sine of the angle of refraction. So that, knowing in any case the direction of the ray before impinging, we can accurately determine the direction in which it is transmitted.

(*c*). Observation shows us that all bodies at the earth's surface gravitate or weigh towards the earth. Farther observation, interpreted according to the laws of motion and force, as hereafter explained, prove that the moon gravitates towards the earth, and that the earth and moon, and all the other planets and moons revolving about the sun, gravitate to that body and to one another. It is thus that, by a gradually expanding induction, we embrace the grand conception of a general force of gravity acting mutually between all the bodies of our solar system.

21. We thus see that it is induction which converts our knowledge which is experimental and particular, into that which is ideal and general. *But it should never be forgotten in our physical inquiries that this general and ideal form of truth could have no existence without previous observations and experiments.*

22. WHILE THE INDUCTIVE PROCEDURE IS ESSENTIAL IN ALL PHYSICAL INVESTIGATIONS, IT IS IN MANY CASES ACCOMPANIED AND AIDED BY THE DEDUCTIVE. — Thus the fact that the weight of a given body is sensibly the same at the earth's surface and at small heights above it, is gathered by induction from observations made upon heavy bodies at and near the surface, at various points. But the fact that the descending motion of a body, caused by its weight, is regulated by a particular numerical law, its speed increasing exactly in proportion to the time, is a deduction inferred mathematically from the constancy of the weight at different heights, taken in connection with our primary ideas in regard to force. So, again, by induction, founded on a great number of observations, Kepler ascertained the general facts in regard to the motions of the planets around the sun; but it was through deduction, springing from these facts or laws, and the axioms concerning force, that Newton was able to demonstrate the great law of planetary gravitation. While, therefore, the physical sciences are in their nature portions of *inductive science*, it should be remembered that deductive reasoning must always have an important share in their development.

CHAPTER III.

GENERAL NOTIONS IN REGARD TO PHYSICAL FORCES AND IN REGARD TO MATTER.—CLASSIFICATION OF FORCES.

23. General Idea of Cause and Effect in Physical Science. — The world exhibits to us a constant succession of events or changes. Observation proves that these are connected with one another in a chain of invariable sequence, so that certain phenomena are the uniform antecedents of other phenomena. The contemplation of these as succeeding each other, according to invariable laws, [illegible] rise in our minds to the idea of cause and its correlative effect. But observation, however extended, can do no more than acquaint us with phenomena and with their uniform sequence. It can give us no assurance that the successive events which we view in the relation of cause and effect are bound together by any necessary connection. Yet such is the constitution of our minds, that a feeling of this connection seems to form an essential element of our idea of Cause. Hence it is that a given event or state of things is regarded by us, not merely as preceding, but as *producing*, that which immediately follows. We look upon the cause of any effect as something without which the effect could not have taken place, and something, the action of which could not fail to give rise to the effect.

24. General Idea of Force and Power in Physical Science. — The idea of cause is of course applicable to all kinds of agency, material and immaterial. But in connection with physical actions and changes, it becomes synonymous with the term Force. Our conception of Force seems to be clearly traceable to the muscular efforts of which we are conscious in our endeavors to grasp and move surrounding bodies, and has its origin in the experiences of our infancy. In this voluntary activity, we are at the same time sensible both of a determinate amount and direction of Force. When, in our subsequent observations, we see one body pressing against another, or impelling it into motion, or resisting and

destroying its motion, we regard it as exerting a species of activity, which, like our own voluntary exertion, is the cause of the pressure, motion, or resistance, which we witness.

25. By a similar procedure, the idea of Power, which originally is only our consciousness of ability to exert force, is made applicable to inanimate objects, and we consider one body, as having the power to move or to resist another, although it may not be in a position to do either. A power in this sense is regarded as having continued existence, and becomes a cause only when effects or changes are produced. The power by which a stone is made to descend is constantly present in the stone and the earth; but stones are not always descending. A horse in a mill is the power which [illegible] motion to the machinery, but it is only when he moves that he causes the mill to move. In this view force may be considered as the exertion of a power in a determinate quantity, as when we speak of the power of a running stream, and of the force with which it impinges upon the floats of a mill-wheel.

26. It must be kept in mind that the terms Force, Power, and Action, are, in strictness, applicable only to the relation between a living voluntary agent and the results of his activity, and that we use them *metaphorically*, in applying them to express the relation of physical cause and effect. Yet it is manifest that no error can arise from the application, because all agree in conceiving the voluntary activity of a sentient being as something distinct from the forces which material bodies exert on one another. When we say that a magnet draws a piece of iron, a stream impels a mill-wheel, or a weight presses upon its support, we are in no danger of confounding the forces which we conceive to be exerted in producing these effects, with the volition of a man who draws, impels, or presses an object.

27. General Idea of Matter. — By matter we mean whatever occupies space; in other words, whatever is extended and impenetrable. We conceive of space in the abstract, as having extension, or more properly as being synonymous with extension; but when to space is added physical resistance, or impenetrability, it becomes body or matter.

The same early experience which gives us the idea of force, gives us also the idea of matter. The child extending its hands, to seize, press, or move contiguous objects, finds its efforts unexpectedly opposed, and thus first becomes sensible of something external or *resisting*. By a further exercise of its muscles it explores the surfaces and outlines of objects, and thus obtains the *idea of space* or extension. This and the idea of resistance are ever after associated in the conception of body or matter.

28. The feeling of resistance, and that of voluntary effort, or force exerted, have their seat in what is called the muscular sense, through which also we obtain the idea of extension. The sense of voluntary force springs up at the very moment that the sense of resistance is exci[illegible] resistance is felt to increase as the force we appl[illegible]s, and to wane away as it is withdrawn. Indeed th[illegible]y idea of muscular effort presupposes some body against which it is exerted, that is, some resistance opposed to it. *Thus our ideas of matter and force originate together, and are inseparably associated.*

29. OF ATOMS, PARTICLES, MOLECULES. — By cutting, crushing, and other mechanical means, it is possible to divide a given mass of matter into a great multitude of parts, and these again into still smaller subdivisions. But however minute each portion may be at the limit which human art can reach, we are still compelled to conceive of it as having length, breadth, and thickness, and we may, therefore, imagine it to be still further subdivided, and this continually and without end. Thus, in our conception, the space occupied by matter is infinitely divisible. But it does not follow that the matter itself is divisible without limit.

Philosophers, in general, maintain that there is a limit to the subdivision of matter, and that supposing the process of division to be indefinitely continued, we would finally reduce the mass to a great multitude of very minute parts, each physically incapable of further separation. These ultimate and indivisible parts are called ATOMS.

In c[illegible]ain physical theories the atoms are regarded as having specific forms, characteristic of the different kinds of matter. These, by their symmetrical union, are supposed to give rise to the regular figures assumed by crystalline bodies, as in the case of rock crystal, the beryl, the diamond, and others. But the form, and even the existence of finite indivisible parts or atoms, admit of no direct proof, and must be viewed as entirely hypothetical.

30. The term PARTICLE is used to signify one of the indefinitely small parts of which bodies are made up without reference to the question of infinite divisibility. These may be regarded as of various orders of complexity. Thus the ultimate particles or atoms of a body would form the very last results of mechanical analysis. But we may conceive these to be arranged in small groups or secondary particles, and we may imagine a mass to be made up of such groups, united [illegible] another by forces springing from the internal constit[illegible] of each, and from their mode of aggregation. Such a complex particle or group of ultimate particles, may be termed a MOLECULE. The molecules thus constructed may again be arranged in clusters, so as to form a still more complex unit, or molecule of a higher order, and so throughout an indefinite series. It is obvious that in this mode of conceiving of the structure of bodies, *we may have an infinite variety of aggregates framed from the same ultimate particles, merely by varying the character and arrangement of the molecules.*

31. Without referring to the ultimate constitution of matter, it is convenient to distinguish between what are called the integrant and the constituent particles of bodies.

The *integrant particles* are the smallest parts of a body, which, if separated from the rest, would retain the mechanical properties of the body unchanged. They are of like nature with the mass. Thus a mass of marble is made up of integrant particles of marble. A lump of brass of integrant particles of brass.

The *constituent particles* are the unlike particles, which, by their union, form the integrant particles of compound bodies. Thus, the particles of carbonic acid and lime, which form the compound body marble, are constituent particles, and are contained in each integrant particle of the marble. So the particles of copper and zinc which unite to form brass are constituent particles.

32. Distinction of Matter into Ponderable and Imponderable. — All the substances with which we are most familiar possess weight. This we know is true of solids and liquids, and although at first view it might seem not to apply to the air, careful experiment shows that this and the large class of analogous bodies, called gases, have the same property in a less degree. Of ponderable bodies the heaviest known is the metal platinum, the lightest is hydrogen gas.

But the property of heaviness or weight is not essential to our idea of matter. The weight of a given mass varies with its position. It will be proved hereafter, that a body weighing 1 lb. at the earth's surface, would, if carried to the distance of the moon, weigh only the $\frac{1}{3600}$ of 1 lb., or less than 3 grains, and yet the quantity of matter, in the mass would remain the same.

The phenomena of heat, light, electricity, and magnetism, have led philosophers to the belief that there exists imponderable matter, of one or more kinds, associated with ordinary matter, and that this is the immediate cause of these phenomena. Hence, heat, light, electricity, and magnetism, are usually spoken of as imponderable agents.

33. Physical forces are in strictness not permanently inherent in masses or their particles. — Powers or forces are often spoken of as inherent in bodies, and when properly understood this language is unobjectionable. But it should be limited simply to the conception, that, under certain conditions, the body will exert force, or will act upon other bodies, and it ought not to be made to imply, except in a merely figurative sense, that the force is continually present, or that it is something belonging to the body equally at all times and in all positions.

We see that a hammer acts on the nail which it drives, and we conceive that it exerts force to produce this effect. But when at rest, or even when moving, it exerts no impelling force. It must be brought into apparent contact with the object in order to drive it.

When a bomb is placed upon its charge of gunpowder, the powder exerts no force to impel the ponderous mass. But if a match be applied, the explosive mixture instantly expands, and in doing so acts with an enormous moving force, which projects the bomb with great velocity into the air. It is obvious that this force is brought into existence by the chemical transformation of the gunpowder, the solid grains of which, previous to this charge, were no more endowed with it than so many grains of sand.

34. Force can in no case be regarded as present, until the conditions which are necessary to the action are present. Action is essentially included in our conception of force. When the proper physical conditions are continued, the action or force is continued; when they are destroyed, the force ceases. By attaching a weight to the hook of a spring balance, the extended spring is made to exert an upward force of recoil which supports the weight, and it continues to act thus as long as the weight remains suspended from it. But when this is withdrawn, and the spring resumes its ordinary length, the force thus evoked ceases to have any existence.

35. *We are not, then, to regard force as something inherent and ever-present in a body, whether there be action or not, but as an activity of the body, which arises or ceases according to the physical conditions in which it is placed.*

36. Of Force viewed as a tendency.—In its most general sense, the term Force signifies any cause which produces or tends to produce a physical change in bodies or the parts of bodies.

The actual production of motion in bodies is not essential to our conception of Force, but merely the capacity or tendency to produce such change. We are conscious of the exertion of Force when we support a weight by a hand extended under it. So, when this weight is placed upon a table, and there supported, we conceive that the parts of the table directly below it exert an upward force equivalent to that which our volition previously imparted to the hand. When we press upon a spring so as to bring it to a certain degree of tension, we are conscious of using a determinate amount of force; and so, when a weight resting upon the spring causes the same effect, we conceive the weight to act with a like force of compression upon the spring. Thus the body supported, and that which it supports, are both regarded as exerting force. In these cases we see that the bodies continue to act upon one another, although there is no motion on the part of either. The supporting force of the table, and the downward force of the weight, are directly opposed and balance one another; and the same is true of the recoiling force of the spring and the compressing action of the weight.

37. If, in the second of the cases just cited, we suppose the supporting spring to be withdrawn, the weight will descend; if we suppose the weight to be lifted off, the spring will recoil. It is obvious, from our mode of conceiving of the forces exerted by the weight and spring respectively, that although they produce no motion when thus opposed to each other, they constantly tend to cause motion, that is, each so acts, that if the other were absent it would produce motion, and this is all that is meant by ascribing to a restrained force an effort or tendency to cause motion.

38. Of the *Equilibrium of Forces.* — In these illustrations we have obvious examples of the *equilibrium of forces.* This term is applied to forces which counteract or balance each other, and thus retain the body or point on which they act, in its original state of rest or motion. If we suppose the mass to be at rest when the forces are applied, it is evident that their equal and opposite actions cannot disturb its quiescence; and again, if we suppose the mass to be in motion at the time the forces begin to act, their balancing action cannot in any degree modify the direction or velocity of its movement.

39. Of *the Mutuality of Actions*, or action and reaction. We have seen that our primary conception of force, as suggested by conscious effort, is unavoidably associated with the idea of resistance, or opposing action. Experience invariably connects a reaction with the muscular force we employ, and the feeling of this reaction becomes greater or less, in proportion as our conscious effort augments or diminishes. Thus, when we push against a wall, or draw a load by means of a rope, we feel that for every change in the force with which we push or pull, there is a corresponding change in the reaction which opposes us. So, in the preceding examples, when a weight rests upon a table, or is supported by an elastic spring, we feel that every increase in

the pressure of the weight must be accompanied by an equal increase in the reaction by which it is supported, for the two opposing forces are in equilibrium. In this way, it would seem that our primary conception of force would lead us to consider reaction as an invariable attendant on action, and as being both equal and opposite to it.

40. But this truth is rather suggested, than clearly proved by our simple and early experiences of force. To establish it as a general law, we must proceed inductively, testing it by a variety of observations embracing the actions of bodies, remote from one another, as well as those contiguous actions from which our first conceptions of force arise. The evidence of this principle will be more fully presented hereafter, when we treat of the Laws of Motion.

41. FORCES ARE DIFFERENT IN KIND. — Enumeration of Forces. In the progress of our inductions, each distinct class of phenomena leads us to the conception of a distinct kind of force. This results directly from the axiom involved in our very idea of cause, viz., that the cause is proportional to the effect, or that the cause must be such in quantity and kind, as to produce the effect. In this way we are led to regard, as so many distinct agencies, —

The FORCE OF GRAVITY, by which the great masses of the solar system, and the parts of which they are composed, mutually tend towards each other;

The FORCE OF COHESION, which resists the attempt to separate the parts of a solid mass;

The FORCE OF ELASTICITY, by which a steel spring or a mass of air reacts when compressed;

The FORCE OF COMBINATION, or affinity, which causes the particles of unlike bodies, under certain conditions, to unite, and which holds them together when combined; and

The FORCES OF HEAT, LIGHT, MAGNETISM, AND ELECTRICITY.

42. As our inductions enlarge, through a more various and exact observation of the phenomena, we discover that classes of facts previously referred to distinct forces, are capable of being united in a single group, and referred to a single kind of force. This has already, to some extent, been the result of a comparison of the phenomena of heat and light, and of those of magnetism and electricity; and it is

reasonable to suppose that still higher and more perfect generalizations, in regard to physical forces, are destined to be attained by future investigation.

43. *Classification of Forces into* Mechanical, Chemical, and Vital Forces. — One of the most obvious modes of grouping physical forces, is that in which they are arranged under these three heads. This classification, founded on the different kinds of phenomena they respectively produce, is not only convenient in treating of them, but must be esteemed philosophically correct, until, by higher inductions, we are able to unite them all under some one common principle of force.

44. Of Mechanical Forces. — In the science of mechanics, as already defined, the only phenomena to be investigated are those of Equilibrium and Motion, and hence the forces with which we are concerned in this branch of physics, are simply the agencies or causes which change, or tend to change the condition of bodies, or their parts, in regard to rest or motion.

We may, therefore, define mechanical force to be whatever tends to cause or to destroy motion among bodies, or the parts of bodies.

45. Of Chemical Forces. — The facts of Chemistry are distinguished by other and more complex characters, such as changes of color, density, form, taste, odor, and other properties, which, while they certainly involve motions among the intimate parts of the bodies affected, are not, like mechanical changes, distinctly traceable to these motions. Still, the only idea which we can form of the causes which produce these changes, is that of certain forces or actions among the particles, by which they are made to unite and remain combined, or to separate from one another and assume new arrangements. In this view, chemical forces are not inherently distinct from mechanical forces; and although for the present it is proper to classify them apart, we may anticipate from future research accumulating proofs of their identity. An approach to this has already been made in the determination of the mechanical force evolved by the chemical action in a galvanic cell; in other words, the mechanical force which is produced by the chemical

action between an acid and a metal while they are combining; and again in ascertaining the mechanical equivalent of heat.

46. Of Vital Forces. — In the phenomena of living beings treated of in Physiology, we may readily trace the operation of mechanical and chemical forces, such as weight, cohesion, elasticity, and chemical affinity. But along with these we discern other agencies, which experience shows to be peculiar to living matter; and these we call vital forces. Such, for example, are the forces determining the growth and development of each elementary cell and of the whole living fabric, the secretion, circulation, and other functions; in all of which ordinary mechanical and chemical forces are also essentially concerned.

47. A living body, viewed simply as matter, is distinguished from common matter by that peculiar structure or arrangement of parts which is called *Organization.* This consists mainly in a complex system of minute cells and tubes, by which the body is enabled to appropriate from without substances suited to its growth and development, and to subject the matter thus received to various mechanical and chemical changes.

48. It will hereafter appear, that the forces exerted by a number of particles in a group, either towards each other or some external body, are capable of endless variety, according to the grouping or arrangement of the particles. Now, as in *organization* we have a grouping of parts which, although varying in different living forms, is quite distinct from the aggregations of ordinary matter, it would seem to be a philosophical inference, that those peculiar forces of living masses which we called vital forces, are the product of this organic arrangement; and that thus *vital forces are the mechanical and chemical forces proper to matter in a condition of organization.*

49. All Forces reducible to Attraction and Repulsion.— However various may be the phenomena of physical change, all the forces in nature, whether operating between masses or particles, whether mechanical or chemical, are reducible to two opposite kinds of action, viz., Attraction and Repulsion. By the former, masses or particles are caused to ap-

proach, or if not movable, to tend towards each other. By the latter they are made to recede, or to tend away from each other.

50. Of Molecular or Corpuscular Forces. — The forces which, according to different physical conditions, operate between the particles of bodies, either as attractions or repulsions, are called Molecular or Corpuscular Forces. When it is an attraction acting between the integrant particles of a mass, it is called *Cohesion;* when between the particles of adjacent surfaces, it is called *Adhesion.* Thus it is the force of cohesion which resists the effort to separate one part of a plate of glass from another; and it is adhesion which resists the attempt to separate one glass plate from another against which it has been pressed. When the molecular force operates between the constituent or dissimilar particles of a compound body, it is called Heterogeneous Attraction, Attraction of Combination, or *Chemical Affinity.*

51. Of Polar Force, or Polarity. — Under certain conditions we find that some bodies exert opposite forces at their opposite sides or ends, attracting by the one and repelling by the other. The points thus opposed in their action are called Poles, and this twofold exertion of force is called Polarity. The action of polar forces is best illustrated by magnets. Let *A* be a compass needle or other magnetic bar, suspended horizontally, and let *B* be a similar needle or bar which is held in the hand, (Fig. 1). *N* and *n* denote the extremities which, in the horizontal position of the needles, would spontaneously turn towards the north — and these are called the north magnetic poles; while the points *S* and *s*, directed the opposite way, are called south magnetic poles. If now we bring the north pole *n* near to the south pole *S* of the movable needle, we find that the latter is attracted by it; but if we present the south pole *s* to the same pole *S* of the movable needle, we see that it is repelled. A like result will be observed in comparing the actions of *n* and *s* on the north pole *N* of the movable needle, except that the actions are reversed, the pole *s* attracting *N*, and the pole *n* repelling it.

CHAPTER IV.

OF THE VARIOUS BRANCHES OF MECHANICAL PHILOSOPHY AND THEIR CLASSIFICATION.

52. All the principal subdivisions of Mechanical Philosophy may be included under the two following heads, —

MECHANICS OF PONDERABLE MATTER.
MECHANICS OF IMPONDERABLE MATTER.

The Mechanics of Ponderable Matter divides itself into the science which treats of the mechanical actions of masses or aggregate forces, and that which considers more specially the actions of particles or molecular forces. The former may be termed the Mechanics of Aggregates; the latter Molecular Mechanics.

53. The following Table presents a classified view of the different branches of this part of Mechanical Philosophy.

Mechanics of Ponderable Matter.

Mechanics of Aggregates.		*Molecular Mechanics.*
Mech. of Solids.	Statics,	Theory of the Elasticity, Strength and Tension of Materials,
	Dynamics.	
Mech. of Liquids.	Hydrostatics,	Theory of Resistances,
	Hydrodynamics.	Capillarity and Endosmose,
		Theory of Waves,
Mech. of Airs.	Aerostatics,	Acoustics.
	Aerodynamics.	
Physical Astronomy.		

Statics treats of the *Equilibrium* of Forces acting upon a point, or a coherent system or mass.

Dynamics considers the effects of the Forces as producing a Modifying *Motion.*

Hydrostatics treats of the Pressure and Equilibrium of Liquids.

Hydrodynamics or Hydraulics, of their Motion and Moving Power.

Aerostatics has reference to the Pressure and Equilibrium of Airs, and Aerodynamics to their Motion and Moving Forces.

In Physical Astronomy are investigated the laws which govern the motions and mutual actions of the celestial bodies.

The Theory of Elasticity, Strength, &c., treats of the molecular forces exerted in machinery, bridges, buildings, &c., and forms the scientific basis of the arts of construction.

The Theory of Resistances includes the laws of Friction, and the laws of Resistance, in the air and in water.

Under the head of Capillarity are included the phenomena of the ascent and depression of liquids within tubes of very small bores, and a variety of other results of the molecular actions of liquids in connection with surfaces. Endosmose treats of the penetration of liquids and gases through porous materials and membranes, and of their mechanical penetration the one by the other.

The Theory of Waves investigates the vibratory motions of the molecules, which by their combination produce waves in solids, liquids, gases, and the imponderable matter of heat, light, &c. This branch of mechanical philosophy belongs as much to the mechanics of imponderable as that of ponderable matter. It is in fact the basis of the wave-theory of light, which is the most generally accepted explanation of the phenomena. It is also the foundation of

Acoustics which treats of the waves or vibrations in solids, liquids and gases, which are productive of *sound.*

54. The Mechanics of Imponderable Matter is divided into

Thermotics, or the Phenomena and Laws of Heat.

Optics, the Phenomena and Laws of Light.

Electricity, those of Electrical Forces.

Magnetism, those of Magnetic Forces.

55. The doctrines of Statics and Dynamics are deduced from a few axioms and simple observations in regard to matter and forces, by geometrical reasoning, and are hence, when taken together, denominated RATIONAL MECHANICS. This part of the subject is in a large degree deductive. But in the application of these principles to the phenomena of nature or the action of mechanism, it is necessary to take account of various agencies, such as friction, resistance of the air, and other molecular actions, the laws of which can only be inferred from multiplied observations. Such is largely the case in the mechanics of liquids and of airs, as well as in the other departments of the subject which have been mentioned. Each branch of mechanical philosophy has thus its own peculiar inductions or laws, directly founded on observation; while to a certain extent the principles of Rational Mechanics are applicable to all.

CHAPTER V.

ELEMENTARY IDEAS IN REGARD TO MOTION AND MECHANICAL FORCES.

56. MOTION is a progressive change of position in some space which is either at rest or conceived to be so. Absolute rest is a condition unknown to us. The earth having a motion of rotation on its axis, and of revolution around the sun, and sharing with the whole solar system in a vast orbital motion among the stars, it follows that all terrestrial bodies are incessantly moving. But these bodies are in the condition of relative rest, so long as they have no motion relative to one another or to the earth.

The only motions of which we can have cognizance are relative motions. To know the absolute motion of a body would require, not only a knowledge of its motion relative to the earth's surface, but all the motions in space by which the earth is affected, and this, it is probable, is beyond the reach of discovery.

57. Motion is a condition of existence, and not a thing. A ball in motion on a table is the same mass as when it is at rest. In the one case it exists successively in different places relative to the table; in the other in the same place. Hence, in strictness, we cannot be said to *communicate* motion to a body, nor can one mass *transfer* its motion to another. When we strike and impel a ball with the bat, we cause or induce motion in the ball. The movement of the ball is produced by the force developed in the moment of impact, and is not a part of the motion originally belonging to the bat. So when a ball in motion strikes a ball at rest, the force developed by the impact causes the latter to start forward, and the latter either to relax its speed or to come to rest.

58. *Thus in the mutual action of bodies, motion is not transferred from one to another, but is produced, modified, or destroyed.* The original motion is the means of bringing the two balls into a position in which they can act upon one another, that is, into apparent contact; and the force due to this proximity is what then imparts motion to the one, and destroys or changes that of the other.

59. A body in motion does not exert force simply in virtue of its motion. Such a body is no more endowed with permanent force than when it is at rest. If we imagine a single body in infinite space, we cannot attribute to it present force, however rapidly it may move. But when by its motion it is brought within the sphere of action, connecting it with other bodies, then this motion, although not itself a force, becomes the means of bringing forces into play between it and the other masses.

60. The rate of motion is expressed by the term VELOCITY. The length of linear space described uniformly in any unit of time, is the measure of the velocity. The unit of time commonly adopted is the second, and the unit of space the foot. Thus a body moving through 100 feet in 1″, is said to have the velocity of 100.

When a body or point passes through equal spaces in equal successive portions of time, the motion is said to be *uniform*. That is, uniform motions are such as preserve their velocity unchanged.

When a body passes over greater or less spaces, in equal successive intervals of time, the motion is called variable. In the former case it is said to be *accelerated*, in the latter *retarded*.

61. The term MOMENTUM is employed to express the product of the mass by the velocity. If three equal bodies describe equal spaces in a given time, their velocities will be equal, and so will their momenta. Unite two of these bodies into one, and let the single and double body pass over equal spaces in a given time. The momentum of the latter will be twice that of the former.

A cannon-ball, whose mass is 24, is impelled with a velocity of 1500 feet per second. The momentum of the ball is 24 × 1500 = 36000. A railway car weighs 2400 lbs., and is moved at the rate of 3 feet per second. Its momentum is 3 × 2400 = 7200, or twice that of the cannon-ball.

62. By a figurative use of language, the term momentum is commonly explained by the phrase *quantity of motion*. But as motion is mere change of place, it is the same in degree or quantity, whether a ton or an ounce be moved, provided the velocity be the same.

63. Mechanical forces are conveniently classed under two general heads, viz., IMPULSES and PRESSURES.

An IMPULSE is a force of insensible duration.

A PRESSURE is one which continues to act for a perceptible time.

We have an example of the former, in the blow of a hammer driving a wedge; and of the latter, in the force which the same hammer exerts while resting upon the surface after the impact. In this example the impulse is seen to be a moving force, while the pressure is balanced by an equal force in the opposite direction, and is therefore quiescent.

64. But this is not necessarily the condition of action of either kind of force. Two impulses in opposite directions may produce rest, as when two equal balls of clay come together with equal velocity; and a pressure may produce motion, as when a man presses forward a railway car, or a horse draws a boat upon a canal.

When a ball is suspended by a string, its weight is balanced by the supporting force of the string, and is thus a quiescent pressure. But as soon as the cord is cut, the same pressure continuing to act draws the body downwards with an accelerated motion, and thus becomes a moving force.

65. **An Impulse, although of short Duration, is not instantaneous, either in generating or in destroying Motion.** — The velocity of an arrow is due to the *continued* action of the bow-string upon it as the bow expands. Hence, the longer the bow, the greater the distance through which it is impelled; because in virtue of the greater bending, the string is made to follow the arrow through a longer space. So the rapid motion of a cannon-ball, when shot from a cannon, although produced in an insensibly small time, is accumulated by the successive action of the expanding gases, as the ball is urged from the breech to the muzzle of the piece. At first its motion is slow, and in the interval between the beginning of the motion and the escape from the mouth of the cannon with its final velocity, it is obvious that the ball passes through every intermediate stage of movement between quiescence and this velocity.

Again, when a ball is shot into a thick and immovable mass of wood, or into a wall of brick, or earth, or stone, although its motion is destroyed in an insensibly short time, we see that this change has been produced by degrees. For the ball penetrates the material to a certain depth before it is stopped; and as the obstruction to its progress begins as soon as it strikes the surface, it is clear that this resistance continues to act as long as the ball retains any of its motion, and that this continued action is what brings the ball to rest.

66. In these cases it cannot be doubted that *more or less time is consumed*, since the impulse includes the action of the force through a measurable distance of motion. But it is not less true of all other examples of impulsion. In all cases of collision, bodies suffer a temporary or permanent change of form. Thus a ball of lead allowed to fall upon a hard surface, will be permanently indented. A ball of steel falling upon the same will be flattened for a moment, and will again recover its original shape. Such a change of configuration implies a motion of the particles through a

finite space; and this cannot take place in an instant, but requires a finite time, and through this time the impulsive forces are acting.

67. From what has been said, we may conclude that *impulses are in strictness not distinct from pressures, but are only pressures which last for an insensibly short time.*

68. Since any finite time may be regarded as a succession of instants, so any pressure may be looked upon as a continued succession of instantaneous actions, and the same view would apply to impulses, as above defined. But the term Impulse is not unfrequently used to indicate an *instantaneous action;* and in this sense it is obvious that all pressures, whether of prolonged or insensible duration, are reducible to a succession of impulses. Thus, when a ball hangs by a cord, its weight or pressure downwards is a succession of impulses following one another without interval, and each of these is balanced by an equal impulse of the supporting force. When the cord is cut, the successive downward impulses add each to the effect of the preceding, and thus augment continually the velocity of the descent. So when a hammer drives a nail, or a hard surface arrests the motion of a bullet, the effect involves a succession of such impulses. But we should be careful not to confound these elementary actions with what are practically the forces of impulse of the hammer and bullet. These latter, as we have seen, are estimated by the aggregate and not the single action of the instantaneous forces.

69. Measure of Quiescent Pressures, or Balanced Pressures at rest. — The simplest mode of comparing two quiescent pressures, is, by their effect upon a spring balance. This instrument (fig. 2,) consists of a steel spring fixed in a frame, and connected by some simple mechanism, with an index movable over a graduated scale, the divisions of which have been previously determined by experiment. Suppose it were required to compare the pressure exerted by a man at a dead pull, (fig. 3,) with that which a weight suspended, (as in fig. 4,) produces in one segment of the rope *B C.* Having placed the spring in the line of the dead pull, so that the whole pressure of the man shall act upon it, we note the position of the index. We

then attach the spring to the rope *B C*, so that it shall bear the whole force which acts in this line, and in like manner note the place of the index on the scale. We will thus have in pounds the measures of the two forces which were to be compared. By this or some similar arrangement, it is obvious that any quiescent pressures may be measured in pounds or such other units as we may prefer. Instruments intended for measuring forces are called DYNAMOMETERS.

70. MEASURE OF BALANCED PRESSURES IN MOTION. — Numerous cases arise in which a pressure is balanced by an equal and opposite force, and yet the mass on which it acts is in motion. Thus, when a horse draws a lighter on a canal with a steady velocity, he exerts a uniform force of traction, the whole of which is expended in overcoming the resistance. If he apply more force than is necessary to balance the resistance, the motion of the boat will be increased; if less, it will be diminished. To measure the force thus applied, we use the spring balance just as in the case of quiescent pressures. We make it form a part of the line of traction, and then, from the position of the index while the force is steadily applied, we learn its value in pounds or other units. It is obvious that a like method is applicable to the tractional force of a locomotive engine, or a steam-tug, or any other balanced pressures in motion.

71. OF THE MEASURE OF MOVING FORCES — WHETHER IMPULSES OR PRESSURES. — As we know of the existence of a moving force only by the motion it produces or destroys, so we estimate its energy by the quantity of motion generated or destroyed. This is not an induction founded on our observation of the action of forces, but springs directly from our idea of force as the cause of motion. It is merely using the effect as the measure of the cause.

If we see a man throw a ball weighing one pound from his hand with a velocity of ten, and another throw the same ball with a velocity of twenty, we at once infer that the force exerted by the latter is twice that exerted by the former. But it is important to remember that in such a comparative measure, we always suppose either that both the forces are instantaneous, or else that they act for equal times.

72. If, in the above example, the second man were allowed to continue his action upon the ball longer than the first, it is obvious that he might accumulate in it two or three times the velocity which the first gave it, without at any one time exerting a greater force than that of the first man. *A force feeble at any one moment of its exertion may, by its continued action, produce in a body any assignable velocity.*

73. Hence, in measuring forces by the motions they produce, we must compare the velocities which they respectively generate by acting on equal masses for equal times. Such a comparison does not consider the entire results of the action of forces, but the relative intensity of the forces at any one instant, or during any equal periods of time. A man, by pushing with a constant force against a railway car for one minute, imparts to it a velocity of two feet per second. But the engine, if attached to the car, would, by its uniform action, in one minute give it a velocity of forty feet per second. It is evident that the moving force of the man is to that of the engine in the ratio of two to forty—or one to twenty. And since each acts with unvarying intensity, the same numbers measure the comparative force of the man and the engine at every moment of their action.

74. Let us now suppose the forces to be opposed to the motion of the mass, so as by their uniform resistance to destroy a part of its velocity. If in equal times the one force destroys a velocity of twenty, and the other a velocity of ten, or if the one destroys a velocity of forty, and the other a velocity of two, we assume, as before, that the effects are proportioned to their causes, and hence we have in the one case the ratio of twenty to ten, and in the other that of forty to two, for the measures of the moving forces. *It thus appears that moving forces are proportioned to the velocities which they generate or destroy in a given mass, by acting uniformly through a given time.*

75. In this comparison of forces, the *uniform action of each is an essential condition.* If either force should vary in its intensity during the allotted time of action, we should in reality have different forces to compare at each moment; and the generated velocities would be proportioned to none of these, but to some unknown average of them all.

76. MOMENTUM THE MEASURE OF MOVING FORCE. — The above mode of comparing forces, conducts us to another and still more important principle. Suppose a moving force represented by f to act upon a given mass, m, for a unit of time, and thereby to generate the velocity, v. If for the mass, m, we substitute one twice as great, $2m$, the same force, f, will in the same time produce only one half the velocity. For we may imagine f to be divided into two equal parts, each acting on one half the mass. Then each $\frac{1}{2}f$ acting on its corresponding m, will generate the velocity $\frac{1}{2}v$, and the whole mass, $2m$, will move on with this velocity.

Now, in order to give to the mass $2m$ the velocity v, it will be necessary to apply the force $2f$. For we may conceive an f separately applied to each m; and as each f will produce in the corresponding m the velocity v, the whole force, $2f$, will produce in the whole mass, $2m$, the velocity v. So to produce in the mass $3m$, $4m$, &c., the same velocity v, we must apply the moving forces $3f$, $4f$, &c. Thus it appears that when moving forces generate equal velocities in different masses, the forces are proportional to the masses upon which they act.

Let us now suppose the force to be such as to produce in the mass $2m$ the velocity $3v$. Since, by hypothesis, the force f produces in the mass m the velocity v, it will require (74) the force three times as great, or $3f$ to produce in m the velocity $3v$. But to produce this velocity in $2m$ will, as just shown, require a force twice as great as this. Hence the force which in $2m$ produces the velocity $3v$, must be of the magnitude $2 \times 3f = 6f$. So the force which in $5m$ produces the velocity $8v$, must have the magnitude $5 \times 8 \cdot f = 40f$.

Comparing the forces with the momenta of the impelled bodies, we have the following numbers:

Force $6f$ . . . Momentum $2m \cdot 3v = 6mv$
Force $40f$ . . . Momentum $5m \cdot 8v = 40mv$.

It is obvious, on inspection, that the forces are in the proportion of 6 to 40, and the momenta generated by them are in the same proportion. As a like conclusion would result in all other cases, we conclude that *moving forces are proportional to the momenta they generate by their uniform action for equal times; that is, to the product of the mass moved and the velocity with which it is impelled.*

77. As examples of this most important method of comparing the intensity of forces, let us imagine a mass weighing 100 lbs. to fall freely, by its own weight, for 1″. The velocity accumulated in it at the end of that time will be 32. Hence $32 \times 100 = 3200$, will be its momentum. Suppose another mass, weighing 200 lbs., placed on rollers on a horizontal surface, to be pressed forward by a uniform force for 1″, and to acquire thereby a velocity of 10. Its momentum will be $10 \times 200 = 2000$. The forces urging the two masses respectively will be in the ratio of 3200 to 2000.

78. Whatever be the expenditure of force requisite to create a given momentum in a small mass, the same force will obviously produce an equal momentum in a large mass, the velocity in this case being less in proportion as the mass is greater. The product $m\,v$ is constant as long as the force f is constant, and when m is increased v must be proportionally diminished.

79. The force required to destroy a given momentum in a moving mass, by acting for a unit of time, must evidently be equal to the force by which, in the same time, this momentum was generated. Thus, to bring to rest a mass of 200 moving with the velocity 10, demands the same expenditure of force as that which was employed in giving to the mass this velocity, the force being now applied as a resistance.

80. THE FORCE WHICH A MOVING BODY IS CAPABLE OF EXERTING, WHEN BROUGHT INTO PROXIMITY TO OTHERS, IS MEASURED BY ITS MOMENTUM. — The mass 200 moving with the velocity 10, and therefore having the momentum 2000, works against the resistance, losing a portion of its momentum at each step, until at length it is entirely consumed. The whole force which it thus exerts in virtue of its momentum, is evidently equal to the force expended in arresting it, and therefore equal to that which impelled it. But either of the latter forces is measured by the momentum of the body. Hence the force exerted by the body is measured by the same.

81. This is what is sometimes called the *Force of Motion*. The fact that the force exerted by a moving body is dependent partly upon its velocity and partly on its weight, admits of innumerable illustrations. A large ship, moving at an almost imperceptible rate, is yet able to crush to pieces any obstacle that intervenes between it and the shore. For, with a small velocity, it has a very great mass, and therefore its momentum is great. A cannon-ball, weighing but a few pounds, when projected at the usual rate, is able to penetrate and tear to pieces walls or other solid objects, even at the distance of miles from the place of discharge. Here, although the mass is small, the velocity is enormously great, and therefore the momentum is large.

CHAPTER VI.

OF THE LAWS OF MOTION.

82. THE fundamental principles included under this name, may be regarded in part as simple axioms, directly dependent on our ideas of force and motion, and in part as general truths founded on observation. In several of the preceding inquiries, we have virtually assumed the truth of one or more of these laws; or have incidentally illustrated them. It is proper now to consider them in the form of distinct propositions.

LAW I. ALL MECHANICAL ACTIONS ARE MUTUAL AND EQUAL: OR ACTION AND REACTION ARE EQUAL, AND IN OPPOSITE DIRECTIONS.

83. We have seen under a former head, (39,) that this mutuality of force is suggested by the same phenomena which impart to us our first ideas of force and matter. Whenever we act upon a body mechanically, we are sensible of reaction commensurate with the force which we employ. But in order to establish the law as an unquestionable and general truth, we must measure the actions of mechanical forces in all the various conditions in which

they operate, as well between remote as between contiguous bodies.

84. This law, in its generality, does not flow necessarily from our ideas of motion and force. We can readily imagine a magnet to attract a mass of iron, and yet the iron not to attract the magnet. We can, in the same way, conceive the earth to attract the moon, without being in turn attracted by that body. The reciprocal action of the forces in these and other cases of action between bodies remote from one another, was first clearly apprehended by Sir Isaac Newton. But it was by a wide induction from observation, that he established this principle as a fundamental law in mechanical science.

Observation proves that all mechanical forces are mutual. The action in one direction is always accompanied by an equal action in the opposite direction. A magnet draws a mass of iron towards it with a certain amount of force, and the iron draws the magnet the opposite way with an equal force. The earth exerts a certain gravitating force upon the moon, and the moon exerts an equal force upon the earth.

85. *The proof of the equality of action and reaction* in these and other cases where motion results, is seen in the *equality of the momenta* imparted to the acting masses. It has already been shown that where the momenta are equal, the moving forces must be equal.

Thus let the magnet and iron be floated on water so as to move freely; then if the mass of the magnet be 20, and that of the iron the same, they will be seen to approach each other with equal velocities. But if the mass of the magnet be 10, that of the iron remaining as before, the magnet will move twice as fast as the iron, thus making the product of the mass into the velocity, that is, the momentum equal for both. Again, if the magnet be held in the hand the iron will approach it, and if the iron be the fixed body the magnet will approach the iron. If both experiments be made at the same distance, the momenta will be found to be equal.

From certain observations to be described hereafter, we can compute the distance through which the moon, if free to fall, would descend towards the earth in one hour, and in like manner the distance through which the earth would move towards the moon in the same time. On comparing these, it is seen that the velocity of the moon would exceed that of the earth in the same ratio that the mass of the earth exceeds the mass of the moon. In other words, they would approach one another with equal momenta.

86. Thus far we have treated the phenomena of mutual action as if in each case two distinct forces were concerned. But although, for many purposes, this mode of viewing the action is the most simple and convenient, it is perhaps more philosophical to regard the phenomena as due to a single force. The magnet and iron are drawn towards one another by the single force of their mutual attraction; the earth and moon are retained in their relative orbits by a force of mutual attraction. When an elastic spring compressed between two balls is allowed to expand, and thus to impel them away from each other, the elasticity which produces both the motions is one and the same force, although for convenience we may regard it as two equal forces acting in opposite directions. In this view of action and reaction, we may consider all mechanical forces as essentially double in their operation; always acting equally in two opposite directions. The law now under consideration, thus interpreted, is equivalent to the following:

Every force, whether an attraction or a repulsion, is exerted in two opposite directions, and generates equal momenta in those directions.

87. Further examples of the Law of Mutual Action:

If two balls, *A* and *B*, having respectively the masses 4 and 1, be forced asunder by the expansion of a spring previously coiled up between them, the velocity of *B* will be four times that of *A*. So if the interposed spring be stretched in the first instance, and thus draw the balls together, their relative velocities will be as before. (Fig. 5.)

This action is strikingly displayed in the *recoil of firearms*. Let a cannon weighing 600 lbs. be suspended in a horizontal position, and let a 12 lb. ball be discharged from it with a velocity say of 1000 feet per second, the cannon will be seen to recoil at the instant of the escape of the ball from the muzzle, and on measurement, the velocity of the recoil will prove to be 20 feet per second. The momentum of the cannon is $600 \times 20 = 12{,}000$; and this is the same as 12×1000, the momentum of the shot. In this case the elastic force of the expanding gases evolved from the powder, acts equally in all directions. But the actions against the sides of the cannon tend merely to stretch the bore, but cannot move the mass, because they balance each other. The action forward against the ball, and backward against the butt of the cannon, is what gives the opposite motions to the ball and ponderous mass.

It results from the principle of equal momenta, that, other things being the same, a light gun will recoil greatly more than a heavy one. A man-of-war, in firing a broadside, is caused to heel towards the opposite side. When she is pursued, guns fired from her stern will accelerate her flight.

88. In the phenomena of impact or collision, we have obvious illustrations of the law of reaction.

Let A and B be two equal balls, which are made to approach each other with equal velocities. At the close of the impact they will both be brought to rest, and if they are inelastic or incapable of recoiling, they will continue at rest. If endowed with elasticity, they will be seen to recede from one another with equal velocities. Here the force due to the proximity of the balls acts equally and in opposite directions upon both, destroying their motion of approach, and in the case of the recoiling balls generating in both equal velocities in the opposite directions. (Fig. 6.)

Let us now suppose the ball A to have twice the mass of B, and their velocities of approach to be equal, as before. If inelastic, we will see them move on together in the direction of the former motion of A, and with a velocity equal to one third of either ball before impact $= \frac{1}{3} v$. Thus the large ball has lost two-thirds of its original velocity, and the smaller one has lost all the velocity which it had, and has, moreover, been impressed with velocity in the opposite direction equal to $\frac{1}{3} v$. The effect of the force of impact on A has been to reduce its momentum from $A \,.\, v$ to $\frac{1}{3} A v$; in other words, this force has destroyed in the ball A a momentum equal to $\frac{2}{3} A v$. The effect of the impact upon B has been to destroy its original momentum $B\,v$, and to generate a momentum in the opposite direction equal to $\frac{1}{3} B v$. Hence the whole effect of the impact upon B is the sum of these momenta; that is, $B\,v + \frac{1}{3} B\,v = \frac{4}{3} B\,v$. But $A = 2\,B$. Therefore $\frac{2}{3} A\,v = \frac{4}{3} B\,v$; that is, the effect of the force of impact upon B is equal to that upon A; in other words, the mutual action of the bodies during collision is equal, and in opposite directions.

89. In the impulsion of a boat by oars, which are moved so as to strike the water backwards, we see that the boat receives the same momentum forward which the oars give to the water in the opposite direction. The same effect is produced by the action of the paddle-wheels of a steamer, or by the propeller revolving at its stern.

90. When a boat tows a ship by means of a rope, the rope is brought into a certain state of tension, and tends constantly to draw its ends together like a stretched spring. This recoiling force continually pulls the ship forward, and tends to drag the boat backward; but the impelling force counteracts this, keeps up the motion of the boat, and thereby the tension of the rope. (Fig. 7.)

LAW II. A BODY CONTINUES IN A STATE OF REST, OR PROCEEDS IN UNIFORM RIGHT-LINE MOTION, UNLESS AFFECTED BY SOME MECHANICAL FORCE.

91. This law is an axiomatic truth essentially included in our idea of force as the cause of motion. Every change in the existing state of a body, whether it be at rest or in motion, necessarily implies a cause of change, and it is to this cause that we give the name of Mechanical Force. When a body previously at rest is seen suddenly to move, or when a moving body is seen to suffer a change in the rate or the direction of its motion, we instinctively refer this change of condition to the action of some force.

92. If we conceive a mass to be at rest any where in space, and at the same time imagine that no force of any kind is allowed to act upon it, we are compelled to believe that it will remain at rest. For by the isolation of the mass we have excluded all cause of change. So if we conceive the body to be moving with a given velocity in a given direction, and entirely cut off from the action of mechanical

forces, we cannot doubt that it will continue in the same direction and with the same velocity as at first; for by our very hypothesis we have excluded whatever could operate to produce a change in the motion.

93. On a superficial view it might be supposed that motion was necessarily a constrained condition of bodies, and that its continuance demanded the continued renewal of the impelling force. According to this conception, a body set in motion with a given velocity in perfectly free space, would of itself slacken in its speed until it ultimately came to rest. But this is to suppose a continual change at the same time that we exclude all cause of change, or else to suppose that the body possesses within itself a force which diminishes and at length destroys its own motion. The former supposition is absurd; the latter is founded on a misconception of the term Force. We have already seen that all force is mutual. A mass of matter alone in space could neither act or be acted on. It could not therefore change its condition by any mechanical action of its own.

94. *Motion is just as much the natural and unconstrained state of matter as rest.* Indeed, as we have already said, it is the only condition of bodies of which we are cognisant. A body in uniform right-lined motion is not in a state of change, but in a specific and constant condition so long as the motion continues unaltered in direction and velocity. This condition once induced in a free mass does not require the continuance of force to maintain it. It is the effect of some previously exerted force, and cannot cease or undergo change unless by some new effort of force upon the body.

95. What we have said may be summed up thus. Every change in the condition of a body as to rest or motion, necessarily implies a cause or mechanical force, and the absence of all mechanical force implies the absence of change; that is, the continuance of the body in rest, or in uniform right-lined motion. This is evidently but another mode of expressing the law now under consideration.

96. But while in the general and abstract view above presented the second law of motion has the force of an axiomatic truth, it is seemingly in contradiction to our daily experience.

The motions which are produced around us do not continue unchanged. The stone which we throw from our hand, or the bullet which is discharged from a gun, move through the air with continually diminishing velocity, and at length come to rest on the ground. A carriage once put in motion on the road does not continue to move at the same rate, but requires for this effect the constant exertion of force by the horses. In numerous instances the direction of the motion also changes as the body proceeds. Thus the ball or bullet, when projected horizontally or obliquely, moves in a curve line, which is continually changing its inclination to the horizon.

97. But when rightly interpreted, the *decay of motion* generally observed, and the change of its direction so often noticed, cannot in any degree shake our confidence in the law of which we are treating. These phenomena are but the effect of mechanical forces which continue after the motion has been impressed, to act either in opposition or transversely to its direction. The resistance of the air through which the projectile is passing continually opposes its motion, and thus constantly diminishes its velocity. The weight of the moving body draws it downwards while it moves forwards, and thus at each moment changes the direction of its motion. So the friction between the wheels of the carriage and the road, as well as that of the axles on which the wheels revolve, presents an incessant opposing force, reproduced at every moment of the motion. Hence the tractional effort of the horses must be continued, in order to maintain the movement of the vehicle.

98. Observation shows that in proportion as we reduce the friction or other resistances to motion, we obtain a nearer approach to absolute uniformity. A ball projected with a certain velocity along a horizontal surface paved with rough stones, will sustain a rapid decrement of velocity, and will come to rest after advancing but a short distance. If impelled with the same velocity over a smooth, Macadam pavement, it will continue its motion for a longer time, and over a much greater distance. Finally, if the experiment be made on a highly polished surface of marble or steel, the ball will proceed for some distance with but slight abate-

ment of velocity, and will continue its motion for a greatly longer time. Although it is not practicable entirely to do away with the friction or resistance of the air in this and other cases, we see that every step we take in reducing these forces, is attended by a nearer approach to uniform motion. We may therefore safely conclude, that, were it possible to secure an entire absence of resistance, the motion once impressed would continue unabated for an unlimited time.

99. Of Inertia. — This term, which is much used in physical science, is employed by different writers to express distinct and apparently opposite conceptions of the properties of matter. According to one of these views, inertia is nothing more than the *indifference of matter to rest or motion.* It denotes by a single word the fact announced in the second law of motion, that a body will continue permanently in the state of rest or motion in which it is first found, until some mechanical force acts upon it. In this sense, the term Inertia is free from objection, and may be advantageously employed.

100. Inertia is also used to signify an *active resistance* on the part of bodies to any change in their condition as to rest or motion, and the term *vis inertiæ*, or force of inertia, is employed to give still greater distinctness to this idea. These expressions are objectionable, because they do not truly describe the physical conditions in view, and may readily mislead the student as to what occurs in mechanical actions. They seem to imply that, by virtue of a certain inherent force, the body at rest is able to keep its position of quiescence in spite of a moving force, and the body in motion is able, for the same reason, to preserve the velocity and direction of its movement.

In one sense, every body, when acted upon by another, exerts a resistance. But this resistance is merely the reaction which accompanies the action, and is equal and opposite to it. It is not a force which affects the motion or rest of the body from which it emanates; but it acts upon the other body. When the mass *A* (fig. 8) impinges upon the mass *B*, the mass *B* reacts upon *A*, and this reaction opposes the motion of *A*, but obviously cannot exert any influence on the rest or motion of *B*. The force acting upon

B produces its full effect upon that body, and not the smallest fraction of it is consumed in overcoming the imaginary *vis inertiæ* or sluggishness of the mass.

101. We have seen (76) that moving forces are measured by the momenta they generate or destroy in a given time. When the force is applied to a large mass, it imparts to it a small velocity; when to a small mass, it produces a great velocity. Yet it cannot be said that the large mass opposes more resistance than the small one to the acting force, for their acquired momenta are equal, and their reactions must therefore have been equal, while the impelling force was at work. From this it would follow that the inertia of the large and that of the small mass are equal.

In the sense of a resistance which consumes the applied forces without effect, it is obvious that the terms *vis inertiæ* and force of inertia are inadmissible.

102. There are other senses in which the term inertia is employed. These are derived more or less directly from our primary conception of force and matter, and are, therefore, to be regarded as philosophically correct. It may be used simply to express *that property of matter, in virtue of which a certain determinate force is necessary to produce a given change in the existing motion of a body.* In this aspect it is but another expression of the principle formerly considered, namely, that the velocity generated or destroyed in a body is the measure of the acting force. To generate or destroy in the mass *A*, the velocity 10 requires a determinate amount of force. To produce the same change of velocity in the mass 2 *A*, will require twice as great a force. In each case the force is consumed in producing the given change of velocity. The term inertia is, therefore, applied to the capacity of bodies to consume force while they acquire or lose velocity; in other words, the capacity of being moved with a certain velocity by a determinate force.

103. According to this view of inertia, it is obvious *that the inertia of a body is, in all cases, directly proportional to its quantity of matter.* Thus, in the example just given, the inertia of 2 *A* is twice as great as that of *A*, for twice as

great a force is expended in giving the same velocity to 2 A as to A.

104. This capacity to consume force belongs essentially to our idea of matter. For by the second law of motion, every change in the condition of a body as to rest or motion, requires the action of a force. But in producing its proper effect, which is the change in question, the force ceases to act; that is, it is expended or consumed. ***We cannot, therefore, conceive of matter ponderable or imponderable***, which does not require force for every change of state, in other words, which has not inertia.

105. By some writers this capacity of bodies to consume force while undergoing change, is itself spoken of as a force, and thus, *Inertia is converted by them into a force.* According to this view, the mass 2 A in the preceding example, having required twice as great a force to cause a given change in its velocity as was required by A, has exerted a double force of inertia. Thus regarded, every mass when acted upon by any external force exerts a force of inertia equal to this force, and in an opposite direction. But this force of inertia is not considered as balancing or neutralizing the effect of the applied force on the mass. On the other hand, the mass moves on with the velocity due to the acting force. Thus, when the mass A (fig. 9) is acted on by the force f, its inertia reacts with an equal force f, at the same time that the mass moves with the velocity proper to the efficiency of f.

106. This mode of viewing the action of forces and the property of inertia, although quite artificial and cumbrous, has been adopted by Poncelet and others, from the facility which it is thought to give in discussing the mechanical problems in which inertia is involved. So far as regards the results of the geometrical reasoning respecting forces, this hypothesis of action cannot lead to any error. For it is evidently quite immaterial, whether the body A (fig. 9) be acted on by the three equal forces f, f_1, and f_2, or by the single force f.

107. As in this sense of the word, the inertia of a mass is always equal to the moving force which acts upon it, it must, like this force, be measured by the momentum gene-

rated or destroyed in a given time. When the mass 50 is impelled to move with the velocity 10, the force of inertia which has been exerted by it is measured by 50 × 10. When the mass 50, already moving with the velocity 10, is compelled to stop by an opposing force, the force of inertia exerted by it is still the same. We thus see that in all such cases, the terms *Momentum and Force of Inertia, have virtually the same meaning.*

108. ILLUSTRATIONS OF INERTIA. — In considering the various actions which are usually adduced as illustrations of inertia, another and equally important physical principle requires to be kept in view. This is the general fact previously insisted on, that a finite time is necessary in order that a force may generate or destroy a finite momentum, in the mass upon which it acts. We have seen that this is true, as well of those impulsive forces which seem to complete their effect instantaneously, as of gravity and other continued pressures.

(*a*). In using the Inertia apparatus, (fig. 10,) a smooth card is placed on top of the column *A B*, and a small weight is laid on the centre of the card. The spring *C D* is then released and gives a quick impulse to the card without moving far enough to touch the column or the weight. The card will be shot from beneath the weight, and the latter will be left resting on the top of the column. In this case the motion of the card is so rapid, that the feeble force of friction which it exerts in sliding under the weight has not time enough to give the latter a sensible forward motion. If we repeat the experiment, using a slow motion of the finger instead of the brisk impulse of the spring, the weight will be carried forward with the card; for now the friction has time to impress the weight with the same motion as the card.

(*b*). When a glass cylinder or globe (fig. 11) containing water, is turned slowly around its centre, the water will not turn with the vessel in a sensible degree until after several revolutions. This will be proved by noting the direction towards which a floating particle or mote points at the beginning and in the progress of the experiment. Here the friction of the revolving glass against the liquid is so insignificant as to require its long continued action to bring the latter into rapid motion. The water is almost in the condition of a mass insulated from the vessel, and therefore unaffected by its force.

When, however, the experiment has been continued long enough, the liquid will be seen to be revolving rapidly with the glass; and if now the rotation of the vessel be suddenly arrested, the water will be seen to continue its motion, until by slow degrees the friction at the sides, reducing its velocity, brings it to rest.

The former case is an example of the permanence of rest; the latter, of the permanence of motion, in the absence of disturbing forces.

(*c*). When a vessel, filled with water, is drawn suddenly along the floor, the water dashes over its posterior side. When the vessel is carried rapidly and steadily forward by a porter, and he suddenly stops or stumbles, the water dashes over the advancing side. In the former experiment we see the tendency to remain at rest, in the latter to persevere in motion.

When a man jumps from a railway car in motion, he strikes the ground with the whole velocity of the car, for in his passage from the car to the ground no force acts upon him, which could in a sensible degree diminish the velocity which he has in common with the car. Hence, by the second law of motion, or the principle of inertia, he retains the velocity until he strikes the ground. So, when the car is suddenly dragged forward, the passengers are made apparently to strike back against their seats, while, in reality, the seats are made to strike forward against them. For not having received the new impulse, the passengers continue in their original state of rest or of slow motion.

(*d*). If a man be standing erect in a boat, when its motion is suddenly arrested by a sunken rock, or other obstacle, he will be suddenly thrown down and forwards; for his feet, by their contact with the deck, are quickly deprived of their forward motion, while the upper part of his body, not having time to receive this check, continues by its motion to move on. Thus a revolving motion is produced, which throws the man down, with his head in the direction of the previous motion of the boat.

It is for the same reason that when a ship, under full sail, strikes upon a shoal, or reef, the men on her deck and spars are often shot forwards over her bow, and her topmasts and even her heavy masts are broken off and thrown down in the same direction.

An effect precisely the opposite of this occurs, when steeples or other buildings are tumbled down by the passage of an earthquake. Here, the rapid horizontal motion of the earth in a given direction, is impressed upon the foundations, or lower parts of the structure; but owing to the great velocity of the motion, these are carried from beneath the upper parts, before the latter have had time to share in the forward movement, and they thus tumble for want of support.

It is upon the same principle, that, in driving home the handle of an axe, or hammer, we either apply a quick impulse to the remote end of the handle, or bring the handle with the tool attached vertically down upon a solid object. In the first case, the motion given to the handle carries it into the eye before time is allowed to impress motion on the head of the tool. In the other method, the motion of the head continues to carry it downwards, after the handle has been brought to rest, and, in virtue of its great momentum, forces the head securely upon the handle, in spite of the resistance of the wood.

109. In all ordinary cases of impulse, the force is applied only to a small part of the body which is struck, and some time is necessary before the remote parts of the mass will feel the effect of the blow, whether it be to increase or to oppose their motion.

(*e*). When a light stick, supported at the ends by two wine-glasses, is suddenly struck in a downward direction, across its middle point, it will be snapped in two without injury to its supports.

A musket-ball, shot with great velocity, will pierce a pane of glass without cracking it, and will penetrate a door, nicely poised on its hinges, without causing it to move. In this case, the transit of the ball through the solid body is so quick as not to allow time for any sensible motion to be impressed on the other particles of the glass, or the door.

But if the ball be simply thrown from the hand against these obstacles, it will crack and crush the glass, and will set the whole door in motion. For in this case, time is allowed for the lateral propagation of motion from the point which receives the blow.

110. The same explanation applies to the still more remarkable fact, of the penetration of very hard bodies, which are at rest, by comparatively soft ones, which are in rapid motion.

(*f*). A candle, fired from a musket with great velocity, will penetrate a board, without being materially changed in shape. In order that the parts of the candle may be displaced among one another, time must elapse. But before the displacement has made much progress, the candle has gone through the door. It is for a similar reason that a wheel of soft iron, made to revolve with great velocity on an axle, will cut through a bar of the hardest steel held across its edge.

111. In all these examples, the property of inertia is manifested as well by the body at rest, as by the moving mass. The former retains its quiescence; the latter perseveres in its motion.

The following instances will illustrate the effects of inertia, where the applied force is made gradually to increase or to diminish the motion of the mass to which it is applied.

(*g*). Let us suppose a man placed behind a railway car, to press against it, and gradually urge it into motion. Assuming the entire absence of friction, and other resistances, the lightest touch of his finger would give motion to the car, however great might be its mass. But the velocity which a given force would generate, in a given time, would be less, in proportion as the mass of the car was greater; and in order to impress the same velocity upon a car of double mass, it would be necessary for him to continue his action for a double time. This is often expressed by saying that the double car has a *double inertia*, and therefore requires a double force, or the continuance of the same force for a double time, to impart to it the given velocity. If now the man continue his pressure so as at length to accumulate a greater velocity in the car, it is obvious that at each step of the acceleration of the motion, the force applied to produce that acceleration is expended or consumed. This action is usually expressed by saying, that the force is expended in *overcoming the inertia* of the mass.

Again, if after a certain velocity has been given to the car, we suppose the man to apply his force in a direction opposite to its motion, he will gradually diminish its speed, until at last he brings it again to rest. Here, for every given decrement of velocity it is obvious that there has been a given expenditure of force; and this also is expressed by saying, that the force is consumed in overcoming the inertia of the mass.

It is obvious, however, that in all this we have merely an example of the principle, that to produce or to destroy motion requires the continued action of some force during a finite time, and that the quantity of motion produced or destroyed is in proportion to the force applied.

(*h*). Again, let us suppose a laborer to seize the winch attached to a very heavy fly-wheel, and by gradual pressure to urge it into rapid motion. At the beginning of his effort the wheel is at rest, and by the law of inertia would remain at rest, did he not apply force to move it. So at each step of the accelerated motion it has a certain specific velocity, which, by the law of inertia, it would retain permanently. But he applies more force, and this adds to the velocity, and, therefore, to the momentum of the wheel. Thus the velocity continues to be increased, so long as he continues to apply force to the mass. This action is commonly expressed by saying that he *continually overcomes the inertia of the wheel.* It is evident, however, that he merely expends his force at each moment in producing additional momentum in the wheel, equivalent to the applied force.

112. When a vehicle or a piece of machinery is in *uniform motion*, overcoming friction or doing its mechanical work, the force which we must use to maintain this velocity is evidently just that which is sufficient to balance the friction or the resistance of the work. The applied force and the resistances being thus balanced, the vehicle or machine is

virtually in the same condition as if no friction or resistance of any kind acted upon it. Hence it will continue to move with a perfectly uniform velocity.

This state of uniform movement under balanced forces, is what is called a *dynamical equilibrium.*

113. In this condition of the moving mass, no part of the motive force is expended in acceleration, and no part of the momentum of the mass is lost in retardation. Hence we have no occasion to consider the inertia of the mass.

But in the first stages of the motion, before the dynamical equilibrium is attained, the force is employed in accelerating the mass as well as in overcoming the friction and other resistances. It is therefore necessary, in estimating the whole force employed, to take account of that part of it which has been consumed in the acceleration of the mass from its state of rest up to the final velocity. In other words, it is necessary to take account of the inertia of the mass.

114. It must not, however, be inferred from this, that the force thus expended in acceleration cannot or does not contribute to the working power of the moving mass. The car which has been thus made to move with a velocity of twenty miles per hour, even after the locomotive is detached, will continue to run on, expending the momentum originally acquired during its acceleration in overcoming the friction and resistance of the air. So, likewise, the fly-wheel, which by slow degrees has been urged into rapid motion, continues its rotation after the motive force has been withdrawn, and, by the enormous momentum due to its great mass and velocity, may be made to execute a variety of mechanical labors requiring great concentration of force.

115. We see by these examples that inertia performs a twofold part. In the first stage of the action of the vehicle or machine, the inertia of the mass consumes so much of the motive power as is necessary to the accumulation of velocity. And in the last stage, or whenever the motion diminishes, the inertia or momentum of the mass furnishes a working power equal to the force which it originally consumed.

LAW III. A BODY MAY HAVE SEVERAL MOTIONS IMPRESSED UPON IT AT THE SAME TIME BY DIFFERENT FORCES, AND EACH OF

THESE IS THE SAME IN DIRECTION AND VELOCITY AS IF THE BODY WERE AFFECTED BY IT ALONE.

116. This is sometimes called the Law of the independence of motions, and was first clearly apprehended and proved by Galileo. The evidence of its truth is less obvious than that of the preceding law. We will consider it under the two heads of the deductive and the inductive proof.

First. *Deductive proof.*

117. This law is regarded by some writers as directly deducible from the ideas we necessarily form of moving forces and their measures. When a body is acted upon by two forces, *P* and *Q*, at the same time, each of them may be regarded as changing the motion which the other, if acting alone, would have produced. The change of motion which the force *P* would thus produce, must evidently be proportional to the changing force, and in the direction of that force. But this change of motion is the very motion which *P*, if acting alone, would have impressed on the body. Thus the body impressed by the two forces has the same motions which *P* and *Q* would have impressed, had each been allowed to act upon it separately. That is, the body so moves in virtue of the combined action of *P* and *Q*, as to exhibit *the whole motive effect of each of these forces.*

Second. *Inductive proof.*

118. All the motions of which we are cognisant are due to the operation of more than a single force, and, therefore, involve two or more directions of movement. While it is obvious that a body cannot move in more than one direction, absolutely considered, at any one moment of time, the single motion by which it is affected may, when viewed in relation to different parts of space, include two or more motions. Thus a man travelling from south-east to north-west, is at the same time moving towards the north and towards the west. This is what is meant by the co-existence of different motions in a body which has been impressed at the same time by different forces.

119. (*a*). Suppose a man to be walking to and fro on the smooth deck of a steamer while she is at rest, or in steady motion. He will find the same expenditure of force necessary to keep up his motion relatively to the deck, whatever

be the direction in which he walks, and whatever the velocity or direction in which the vessel moves.

If the steamer be moving forward at the rate of five miles an hour, and he walks towards the bow at that rate, his actual velocity over the water will be ten miles an hour. But when he walks in the opposite direction, or towards the stern, at the same rate, his velocity in regard to the water will be nothing. In the one case, his effort doubles the previous velocity which he had in common with the deck; in the other, it just destroys that velocity, and holds him suspended, as it were, over the same spot on the water, while the deck moves forward from beneath him.

In both cases the result includes the two motions — the motion due to the vessel, and that due to the muscular effort of walking. In the first instance the man goes forward by walking, and goes forward by the motion of the boat; in the second, he goes backward by walking, and forward by the motion of the boat.

120. (*b*). Imagine a smooth cake of ice (fig. 12) to be floating with a steady motion down a stream; and suppose a boy standing upon it to slide across it in a direction at right angles to that in which it is moving. It is obvious that the boy will be affected by both motions. He will be carried down the stream just as fast as if he were not sliding on the ice; and he will move across it just as rapidly as if he were not floated down. If therefore we assume *a b* to represent the distance through which the ice moves in the same time in which the boy slides from *a* to *c*, his actual motion obliquely across the stream is such as to include *a b* and *a c*. By completing the rectangle *a b c d*, it is obvious that at the end of the given time the boy will be at a point of space represented by *c d*. His motion has in reality been in the line *a d*, which is equivalent to the motion *a b* downwards, and *a c* across the stream.

121. (*c*). A weight dropped from the mast-head of a ship will strike the deck at a point vertically below, although the vessel be in rapid motion, provided her velocity be uniform during the fall. Let *A B* represent the vertical mast of a vessel; and suppose that in the time occupied by the weight in its descent to the deck, the vessel moves from *A*

to *a*. The weight commencing its fall at *B* strikes the deck at *a*, having during the interval descended through a distance equal to *B A*, and advanced through a distance equal to *A a*. We thus find that its motion forward continues at the same rate as that of the vessel, notwithstanding it is falling, and that its motion downwards is as rapid as it would have been had the vessel been at rest, and the weight therefore been without any forward motion.

The actual motion of the mass being that of a body which falls by its weight at the same time that it advances, is exactly like that which would result if the mast were at rest and the weight were thrown horizontally from the summit with the same velocity as that formerly possessed by the ship and mast. Hence, when viewed from the shore, the weight is seen to describe the descending curve *B a*, while to an observer on the deck it appears only to move vertically downwards.

122. (*d*). It is evident that, in virtue of the earth's motion on its axis, every falling body must advance at the same time that it descends. When its descent is from a small height, as from the top of an ordinary building, it will strike the ground at a point vertically below that from which it fell. But when it falls from a very lofty spire, or from the top to the bottom of a deep mine shaft, it will be seen to strike the earth a little to the east or in advance of the point vertically beneath. The cause of this deviation is thus explained.

Let *M N* (fig. 13) denote a portion of the earth's surface, and *A B* a lofty tower, which by the rotation of the earth is carried from the position *A B* to *a b* in the same time in which a ball dropped from *B* descends to the ground. The velocity of the top of the tower is greater than that of its base, in the proportion of *B b* to *A a*. If then we take, on the surface, the line *a d* equal to *B b*, the distance *b d* will measure the excess of the velocity at the top of the tower compared with the bottom. The ball when it begins to fall has the forward motion of the top, and retaining this while it descends, it advances or moves toward the east more rapidly than the base of the tower. Thus, instead of striking the ground at *b*, it strikes it at the point *d*, a little to the east of the vertical. (Fig. 5.)

Experiments of this kind, made in certain mines in Saxony, have shown a marked deviation of the falling mass towards the east.

123. (*e*). When a pendulum swings from west towards east, its motion conspires with that of the ground on which it is placed, and when it swings in the opposite direction its motion is backwards in regard to the earth's rotation. Yet, observation proves that its time of vibration is exactly the same in these and all other directions in which it can oscillate. From this we conclude that the force of gravity which vibrates the pendulum, produces exactly the same effect upon it, whatever be the direction of the pre-existing motion due to the rotation of the earth. In other words, the two motions are independent of one another, and the two forces, that of rotation and vibration, produce their respective effects, each as if the other did not operate.

124. (*f*). On board of a vessel which is moving smoothly and uniformly, we can write, or draw, we can hammer, saw, or plane, we can walk, run, or dance, with no further effort than would be necessary were we on shore. Yet in each of these acts, the special motion we make is merely a modification of the general motion which we and every thing about us have in common with the ship.

The pen with which we write, while it traces the letters on the paper before us, is tracing an invisible line above the surface of the sea, marking, in a united form, the great motion of the ship, and the minute lateral deviations of the pen as it shapes the letters. If we adopt the illustration used by Galileo, and suppose the vessel to make a voyage across the Mediterranean while we are writing, and that, by the action of the waves, she is made to pitch and to roll while she moves on, we shall have an invisible line traced by the pen, which will include all these motions. It will mark the forward movement of the ship, the alternate rising and falling at the bow and stern, the heeling and rolling sideways, and the peculiar movement of the pen over the paper in the act of writing. Thus, in the one complex motion we see included many particular movements, and the force proper to each producing its whole effect irrespective of the rest.

125. (*g*). The principle under consideration is in no way more strikingly illustrated, than by the fact that the motion of any spot on the earth's surface is incessantly changing both in direction and velocity; and yet, a given force operating upon a body at that place, will, at all hours of the day, or periods of the year, impress upon it the same relative velocity.

(Fig.

Let *A B* (fig. 14) represent the earth, and *M N* its orbit around the sun at *S*, and let the arrows severally denote the direction in which the earth rotates once a day upon its axis, and once in a year around the sun. The velocity of rotation proper to any point, *B* revolving in the equator, is about 1500 feet per second. This velocity is known to be perfectly uniform. The velocity with which the earth's centre, *C*, revolves in the annual orbit is subject to slight periodical variations, but may be stated in round numbers at 100,000 feet per second. (?) 90.000. ft. per. Sec.

When a spot on the earth's surface is brought to the position *B*, which is the mid-day point, its rotary motion is backward in comparison with the motion in the orbit, and therefore its actual motion in space is the difference of the two, viz. 100,000 — 1500. But when, twelve hours later, the spot passes round to the position *A*, or the midnight point, the rotary and orbital motions conspire, and therefore the true motion of the place is equal to their sum, or 100,000 + 1500.

The difference between these velocities at *A* and *B* is 3000 feet per second, which is nearly twice the velocity usually imparted to a cannon-ball. This enormous change of velocity is, of course, shared by all bodies on the surface, and yet we know that a given force applied to any mass thus placed, will produce the same motion at mid-day, at midnight, and at all other hours.

So again if we imagine the earth to be at *N*, *O*, or *M*, the motions we have just considered would be in an entirely new direction, but the action of forces at the earth's surface would undergo no change.

126. If we imagine a cannon to be placed above *B*, and pointing towards the east, to project a ball horizontally with a velocity of 1500 feet per second, it is evident that the

velocity of the ball relatively to the earth's centre will be 3000 feet per second. But if the ball be impelled by the same charge of powder in the opposite direction, although it will separate from the cannon as before, with a velocity of 1500 feet per second, its velocity, relatively to the earth's centre, will be nothing.

In both cases, before the ball is discharged, it and the cannon and the adjoining ground are moving towards *E* with a velocity of 1500 feet per second. When the projection is towards *E*, the impelling force adds to the already existing velocity of the ball in that direction an equal velocity, and therefore the motion relative to *C* is at the rate of 3000 feet per second. But when the projection, as in the second case, is in the opposite direction, the force of the charge has merely the effect of destroying the existing motion of the ball due to the earth's rotation, and while the ball is thus arrested, the cannon, and adjacent ground and spectators move away from it, in obedience to the general rotary motion. In the one case, the projectile force doubles the velocity of the ball; in the other, it destroys it; in the one, the velocity produced by it is added to that already present; in the other, by opposition of direction is subtracted from it. But in both cases the mechanical effect proper to this force is the same.

CHAPTER VII.

MECHANICAL CONSTITUTION OF MATTER.

127. Under a former head, some general ideas were presented in regard to the Molecular structure of matter. But a further and more detailed consideration of the subject is now necessary, to prepare the way for a systematic view of succeeding topics. We shall consider the subject under the heads of

1st. The Tenuity or Divisibility, and Porosity of Matter; and

2d. The Molecular Structure of Matter.

128. **Of the Tenuity or Divisibility of Matter.** — Mention has already been made of the great extent to which the subdivision of matter can be carried by artificial processes.

The following Table will serve to show the attenuation of certain bodies, in the form of films, fibres and globules:

Attenuation of Matter measured in parts of an inch.

Films.		Fibres.		Globules.	
Soap bubble at top just before it breaks,	$\frac{1}{4000000}$	Human Hair, about	$\frac{1}{600}$	Human blood, about	$\frac{1}{3000}$
Wings of some Insects,	$\frac{1}{100000}$	Coarse wool,	$\frac{1}{500}$	Of Horse,	$\frac{1}{4000}$
Gold leaf,	$\frac{1}{290000}$	Fine "	$\frac{1}{1500}$	Dust of Lycoperdon,	$\frac{1}{30000}$
Gilding of Wire,	$\frac{1}{2000000}$	Cotton,	$\frac{1}{1000}$	Animalcules Monas,	$\frac{1}{18000}$
Writing paper, from $\frac{1}{500}$ to $\frac{1}{800}$		Flax,	$\frac{1}{2500}$	Vibrio,	$\frac{1}{2000}$
		Silk,	$\frac{1}{2000}$		
		Spider's line,	$\frac{1}{30000}$		

129. A few remarks in reference to some of the objects mentioned in this table, will show more clearly the wonderful tenuity of matter, even in forms which are palpable and apparent.

130. The top of the soap bubble when only $\frac{1}{4000000}$th of an inch in thickness, still forms a continuous liquid envelope of the air. The wire gilding, although only $\frac{1}{2000000}$th of an inch thick, is still so entire a sheet, that the most powerful microscope cannot detect a break or aperture through which the inclosed copper or brass can be seen. What is more, the wire may be plunged in nitric acid, one of the most powerful solvents of these metals, and yet the thin film of gold will prove a perfect barrier to its action.

131. Under the microscope, each fibre of cotton wool appears to be composed of two tubular cylinders, joined together by an interposed band. Each fibre of flax is a bundle of smaller fibres, which are jointed and irregular. The diameter of these is what is given in the table. The thread of the spider results from the expulsion of a peculiar

viscid matter from six teats under the animal's belly. Each of these is pierced by a multitude of apertures, probably not less than one thousand, and from each aperture proceeds a separate fibre; these fibres uniting and adhering to form the complex rope or line.

132. The blood globules are little red bodies which float in the watery fluid or serum of the blood, giving color to the mass. Although called globules, they are in reality flattened or lens-shaped, and are not unfrequently packed together in groups. In all mammiferous animals they are circular. In birds and fishes their form is elongated.

133. In the world of animalculæ we have still [illegible] surprising proofs of the tenuity of matter. Some [illegible] living objects, far less in dimensions than a globule [illegible] blood, are yet provided with organs of locomotion and defence, of nutrition, digestion, and other complex functions. They have their solid sheath or skeleton, composed of pure flint or limestone, and their muscles, tendons and circulating fluids. How inconceivably minute must be the globules which move in currents through their minute vessels and cells, or the elementary fibres of which their muscles are composed.

134. The coloring power of many bodies affords another example of great subdivision of matter. One grain of copper dissolved in nitric acid, will give a distinct blue color to three pints of water. This contains 104 cubic inches, and the eye can distinguish a portion, which is $\frac{1}{100}$th of an inch each way, that is $\frac{1}{1000000}$th of a cubic inch in bulk. Each of these parts must then contain only the $\frac{1}{104000000}$th of a grain of copper.

135. In odors, we have a still more striking illustration of the tenuity of matter. It is said that a grain of musk will continue for several years to perfume the air of a chamber, twelve feet square, and will suffer little or no diminution of its weight. Now each cubic tenth of an inch has sensible odor, and therefore contains some of the musk. But the whole volume of the chamber amounts to nearly 3.000.000.000 cubic tenths of an inch, and the air must of course be often renewed, in the course of so long a time. The weight of musk, then, capable of affecting the sense, is incalculably

small. The same is true of other odors, and of those subtle poisons which infect the air of certain regions, and are yet undiscoverable by the nicest chemical tests.

136. Of the Porosity of Bodies. — Obvious examples of this structure are seen in sponge, loose sand, pumice-stone, clay, most rocks and minerals, wood, bone, and organic textures generally. Some minerals of great compactness are filled with invisible pores, which are sufficiently large to allow water to be imbibed by them in great quantity. This is remarkably the case with the siliceous mineral called hydrophane; which, after being plunged in water, takes up about one-sixth of its bulk of the liquid, and acquires the transparency of glass. The great porosity of wood is shown by the enormous augmentation of weight which a piece of perfectly dry wood will suffer, when thoroughly saturated with water. In this condition, a mass of wood ceases to be buoyant, and will rapidly sink in water. The porosity of rocks, sand, earth and clay are manifested by the penetration of water through these materials, to great distances downwards and laterally, and its consequent supply to springs and artesian wells.

137. Molecular Structure of Matter. — Porosity, as just exemplified, is nothing more than the existence of a multitude of minute cavities in the interior of a solid, and does not necessarily imply that the matter of the body itself is not absolutely continuous, or free from pores. A piece of glass when filled with air-bubbles, is porous, and yet we know that the solid substance of glass is devoid of porosity. These considerations lead us to a question of great interest in physics, viz., Are the integrant particles, or molecules of bodies in absolute contact with one another in the mass, or are they arranged at distances? The answer to this inquiry is readily gathered from the phenomena of mechanical action, and it is this: —

138. *The contiguous particles of bodies are not in mathematical contact with one another, but are placed at distances.*

139. The most obvious objection to this conception of the structure of bodies, arises from the fact that the surfaces of glass, steel, water, and other bodies which are without

pores, have the appearance of unbroken continuity, and the same thing is true of the fresh surfaces developed by fracture, or separation of the mass. But to this objection we reply that the *seeming* continuity of surface is no proof of actual continuity. The particles and their interspaces are far too minute for separate recognition by the microscope. Numerous examples may be given of this appearance of continuity, when there is a wide separation of the parts of the object.

Thus, when a sheet of white paper, dotted evenly with black spots, is placed at a distance from the eye, it appears of a uniform gray color. Yet, when brought nearer, we see that the black spots are separated by wide intervals.

So a surface covered with a mixture of blue and yellow pigments, appears green. Yet there are no really green particles present, and with the microscope we can readily distinguish the little dots of blue and yellow matter, evenly scattered over the surface.

The Nebulæ, when viewed by the naked eye, or through a feeble telescope, present the aspect of light clouds of continuous substance, floating in the sky. But when examined through a powerful instrument, they are seen to consist of multitudes of distinct stars, placed at vast distances apart.

We may therefore conclude that the apparent continuity of a mass of matter is no proof that its parts actually touch one another, and that notwithstanding this appearance, the particles may be placed at distances vastly greater than their own diameters. Indeed, as we shall presently learn, the volume of interspace in many bodies is almost immeasurably greater than the volume of the matter itself.

140. The direct proof of the existence of intervals between the particles of bodies, is founded on the fact that all kinds of matter, solid, liquid, and gaseous, are capable of being reduced in volume, by the application of pressure. This compressibility is shown by air, and the other gases, under very feeble forces; but it is not strikingly exhibited by either liquids or compact solid bodies, unless great pressure is applied.

A mass of common air, contained in a close cylinder of metal or glass, may, by the descent of a piston, be reduced to one-half, one-third, one-tenth, or even one-hundredth of its original volume. Regarding the air as made up of similar particles, each of definite form and dimensions, this reduction in the volume of the mass, leads us inevitably to the conclusion, that, in its usual state, the particles of air are placed at great distances from one another. In this instance it is obvious that the entire space unoccupied by the particles vastly exceeds that which is filled by them.

141. The same reasoning evidently applies to liquid and solid bodies. The compression in either case can only be regarded as a forced approach of the particles of the solid or liquid, and is therefore proof that they were not previously in contact.

142. In the case of water and other liquids, the evidence of this truth may be presented in another shape, founded on the combined consideration of their compressibility and the mobility of their parts.

Let *A B C D* be a mass of water filling a cubical vessel, (fig. 15.) We know that with a light straw we can move the particles of liquid from place to place. If, therefore, it be maintained, that they are in mathematical contact, we are compelled to suppose that they are spheres. For no other form would be compatible with their free motion among each other. Now, let this mass be subjected to gradually increasing pressure. Experiment shows that it undergoes a reduction of volume. But this can only happen in consequence of a flattening of the spheres at the points where they touch, or a general contraction of each sphere equally throughout. The former seems alone consistent with the directions in which the pressing forces act between them. By this flattening it is obvious that the particles would become more difficult to displace; and when, at length, the compression converted the spheres into cubes, the mass of water would retain no trace of its fluidity, but would form an impenetrable and incompressible solid body.

But observation contradicts this conclusion, by showing that when the mass of water is under the most enormous pressure that can be applied, the particles are as freely moveable as before. Hence, the hypothesis with which we started is incorrect. Spherical particles in contact will not account both for the fluidity and compressibility of a liquid; still less will particles of any other form answer the purpose. We must therefore conclude, that the particles of liquids are not in contact.

143. Of the Molecular Forces between contiguous particles. — *The contiguous particles of solids and liquids may be considered as neither attracting nor repelling one another while they are placed at their natural distances.*

When we attempt to separate the parts of a solid body, as a rod of iron or wood, by drawing them asunder, we find our effort resisted, and this resistance we ascribe to an attractive force which we call cohesion. If, on the other hand, we endeavor to press the parts nearer to each other, we encounter a resistance in the opposite direction; and this we ascribe to the mutual repulsion of the particles.

In both of these cases the molecular action does not arise until the distance between the particles has been changed by the applied force. This is clearly proved by suspending the rod in a vertical position, and suddenly hanging a weight to its lower end. In the first moment the weight descends a small space, stretching the rod, and therefore drawing the particles farther asunder. When their separation has reached a certain stage, the mutual attraction of the contiguous particles is sufficient to sustain the load.

In like manner when the weight is laid upon the top of a vertical bar or column, it at first descends and thus forces the particles nearer to one another, which brings in action a repulsive force competent to support the weight.

144. A similar effect takes place when a liquid is compressed or dilated by mechanical means. When a confined mass of water is subjected to pressure, its particles are at the same time brought nearer and made to repel each other. When the action is applied in the opposite way, the particles are removed to greater distances and made to attract.

It thus appears that, under ordinary conditions, the contiguous particles of a solid or liquid mass exert towards each other neither attraction nor repulsion. But as soon as they are drawn farther apart, they begin to attract; as soon as they are pressed nearer together, they begin to repel.

145. The force of cohesion is therefore not a continually acting force, connecting together the molecules of solid and of liquid bodies, but is called into play by any effort which is made to separate the contiguous particles.

The molecules of air and other gases are mutually repulsive under all circumstances. Whether the air be of common density, or compressed or rarefied, it will be found, when brought into an empty space, to expand rapidly on all sides, in virtue of the continued repulsion among its particles. This is the distinctive character of aeriform matter.

146. Of the Restoration of the displaced Particles, or Elasticity. — When any applied force causes the particles of a body to change their distances, they tend to resume their former positions as soon as the disturbing cause is removed. If they have been made to recede, corpuscular attraction will restore them to their former places; if they have been made to approach, corpuscular repulsion will separate them to their original distances.

This is what constitutes the *elasticity* of masses. An elastic string may be extended beyond its natural length, by some foreign force, but when this force is withdrawn, it contracts again. An elastic solid may be variously compressed, either through its whole extent, or in certain parts; but upon the removal of the compressing force, it recovers its former state. It will be shown hereafter that even those bodies which seem to have no elasticity have the power of restoration, in a perfect degree, within small limits of displacement.

147. *Force is not transmitted through a mass, but the molecular forces are successively excited, by the pressure or impulse which we apply.*

If we suppose a row of particles to be arranged in a direct line at their natural distances (fig. 16), and a pressure to be applied to the particle *A*, at one end of the row, in the direction of the line, it will appear to act on them all, and to impel the whole row simultaneously. But this is evidently not the process of nature. The internal force moves only the first particle. This is now within the limit of the mutual repulsion between it and the second particle

B. The repulsion thus brought into play moves *B*. The approach of *B* to *C* brings them within the limit of repulsion, and thus *C* is made to move. In this way the repulsive forces of the particles of the row are successively brought into action, and always in the direction of the applied force, and with an equal intensity.

Now it is obvious that the force which moves the second particle is not the identical pressure which moved the first, nor are any two particles moved forward by the same force, but each exerts its force of repulsion on the next.

148. If the external force act in the contrary direction, tending to draw the particle to which it is applied away from the one next to it, the distance between them being increased, a mutual attraction is excited. This attraction causes the second particle to follow the first. The distance between the second and third is now increased, and they attract. The third particle follows the second. In this way, the mutual force of cohesion causes one particle to follow another throughout the whole line.

149. Strictly speaking, therefore, a particular impulse or pressure cannot be transmitted through a mass. But by its action at a given point, it is the means of bringing successively into action the molecular forces of the contiguous particles. Thus motion is generated in every particle, in the same direction, and the whole mass advances in the line of the applied force.

150. ALL CONTIGUOUS ACTIONS ARE IN REALITY ACTIONS AT A DISTANCE. — By contiguous actions are here meant those which occur between bodies in apparent contact; as when a ball of ivory impinges upon a hard surface, or when a heavy weight is supported on a table.

As our ideas of force spring originally from the impressions caused by grasping or touching the bodies near us, it is natural that we should consider contact or close proximity as necessary, in all cases of mechanical action. Hence it is not surprising that for a long time it was a received dogma in the schools, that one body could not act upon another, unless it was in absolute contact with it. But a larger observation and wiser interpretation of natural phenomena has served to show that some of the most important of these effects arise from the mutual actions of bodies which are separated by great distances. All the phenomena of planetary attraction are of this kind. The fall of a stone to the ground, is a familiar instance of action at a distance; for it is by the attraction of the earth that it is drawn downwards. The attractions and repulsions of magnetic and electrical bodies present a variety of effects, all of the same class.

151. This general prevalence of the action at a distance in the great movements of nature, may prepare us to admit its operations in those cases in which there is a seeming contact of bodies.

We have already seen that the contiguous particles of a body are separated from one another by intervals of space, and that, in many cases, these intervals are probably of vast extent compared with the particles themselves. Yet we know, that in all the cases of impulse or pressure, in which force is successively excited between the particles, they are made to attract or to repel one another across the separating spaces.

When by a blow on one side of a football, (fig. 17,) the whole mass of included air is made to move forward, the motion is successively excited in the particles from *A* to *B*, by a very small compression at *A*. Yet we know that this air might be compressed into one hundredth of its volume, and still admit of a further reduction. It is clear, therefore, that the particles of air have acted upon one another while at a distance.

We conclude, then, that the action among the particles, forming any solid liquid or gas, are actions at a distance.

152. Let us now inquire what takes place when one mass presses or impels another with which it is in apparent contact. In such cases, do the acting particles approach until there is absolutely no space left between them, or do *they exert their mutual forces through an insensible but still actual distance?*

The following considerations will serve to show, *that in all the contiguous actions between bodies, the acting surfaces are at a distance from each other.*

153. Since it has been shown that the particles of the most compact and solid bodies, even when subjected to pow-

erful pressure, are not in absolute contact, it might with great force be urged that the surface of contiguous bodies, in the act of pressing or impulsion, are still separated by a small distance. But beside this general argument, we are able to adduce the fact, that in certain cases of contiguous action the distance between the adjoining surfaces has been actually determined.

154. When a lens of glass rests on a polished plate of the same substance, in a horizontal position, the former presses on the latter with a force equal to its weight. From certain optical phenomena exhibited by the arrangement, Sir Isaac Newton was able to demonstrate that the surfaces where nearest together, are at a distance of not less than $\frac{1}{4450}$th of an inch. Yet the lens is supported by the plate, and the plate is pressed by the lens. It is evident that the mutual repulsion between the particles of the plate and lens holds up the lens, and operates downward on the plate with an equal force. The repulsion acts across the interval of $\frac{1}{4450}$th of an inch. (Fig. 18.)

In this case, we see that the mechanical phenomena of contact take place when the surfaces are at a measurable, although small distance. Applying this to all similar cases, we may define physical or practical contact to be that degree of proximity between surfaces which brings them within the sphere of corpuscular action. *The act of touching is not the annihilation of distance between the touching surfaces, but their approach within the limits of mutual repulsion.*

155. The distance between two surfaces in physical contact, can be diminished by external force. It has been found by experiment, that a pressure equal to 1000 pounds to a square inch is necessary to bring the lens of the former experiment so near to the plate as that light shall not be reflected between them. Yet, although apparently forming one mass where they touch, the two surfaces do not cohere but repel one another, with a force equal to the external pressure. Their distance, when nearest, must, therefore, greatly exceed that of the particles of the solid glass. Although not exactly determined, it has been proved by Robison, to be much more than the $\frac{1}{88000}$th of an inch.

If, under this enormous pressure, the particles are not caused absolutely to touch, but are greatly more remote from one another than the molecules which form the solid mass, it is reasonable to conclude, that in all cases of contiguous action, either of impulse or pressure, the corpuscular forces operate through a distance.

156. Thus, when a ball *A* impinges upon another ball *B*, (fig. 19) the action is simply as follows. When *A* has approached within the limit of physical contact, or molecular repulsion, this force begins to operate in both directions, diminishing the motion of *A*, and impressing motion on *B*. As, by the first law of motion, these opposite actions are equal, the momentum generated in *B* is equal at each instant to that which is destroyed in *A*. This mutual action continues as long as the bodies are within the sphere of their repulsion.

157. When the first of a row of equal ivory balls (fig. 20) is struck, it moves through an insensible space and comes within the limit of repulsion of the next. The repulsion being mutual, impels the second ball and stops the motion of the first. The second, in its turn, approaches the third, and the mutual repulsion thus excited arrests the former, and sets the latter in motion with the same velocity. As the motions of the intermediate balls are confined to almost insensible distances, and occupy very little time, the last ball is seen to spring away and the first to come to rest, almost in the instant of the first impact.

158. Of Alternations of Force within the Limits of Apparent Contact. — If two pieces of mirror glass be carefully adjusted to the distance of about $\frac{1}{2400}$ of an inch, their contiguous surfaces will be found to exert a strong attraction, so that the lower glass, even when loaded, may be lifted by the upper. If the distance be augmented to twice that above mentioned, the adhesion ceases. If it be diminished, the adhesion proves less; and, as we have already seen, when the distance is reduced to $\frac{1}{4450}$, or about one-half, the surfaces repel one another. But we know that the particles of glass in either of the masses used are placed at the limit between attraction and repulsion, and also that they are much nearer than the $\frac{1}{4450}$th of an inch. We are, therefore, en-

titled to affirm that two alternations of attractive and repulsive force are actually proved by observation.

159. The successive forces are as follows:—

1st. The Attraction which takes place when the surfaces are at a distance of about $\frac{1}{2400}$ pt. of an inch.

2d. The Repulsion which shows itself decidedly at $\frac{1}{4405}$th, and which continually increases as the distance grows less, down to some unknown limit.

3d. The Attraction of the particles of glass in the mass, which occurs when their distance is a little greater than the natural distance.

4th. The Repulsion of the particles of glass in the mass, when their distance is a little less than the natural distance.

160. That other alternations succeed, as the distances are still farther diminished, is proved by various considerations, among which may be noted the structure of minute organic bodies, and the phenomena of chemical change.

Thus, some animalcules are so small that they could exist between the lens and plate-glass in Newton's experiment, even when these are strongly pressed together. The distance of the two surfaces would afford ample space for their movements. But the complexity of their structure, their cells, canals, integuments, and moving liquids, show that within this limit there must be a force binding the particles together, and again within this narrower space still further alternations.

In chemical compounds, the attractions and repulsions which operate between the constituent atoms of the complex particles, belong to intervals of distance greatly less than those at which the cohesion and elastic repulsion of the molecules are brought into action.

161. *Generalizing these observations, we are led to regard molecular action as entirely dependent on the distances and arrangement of the atoms or molecules.*

The very same particles which in one mode of grouping present the cohesion and immobility of parts, characterizing a solid, in another, form a liquid, and in a third, a self-repellent vapor or gas. By assuming that all matter consists of ultimate atoms, identical in their nature, and that these are endowed with the power of mutual attraction and repulsion, according to the particular intervals of space at which they are placed; we can understand the mechanical cause of many of the actions and transformations of bodies which would else remain unintelligible.

162. By supposing the atoms to be grouped in a certain way, so as to form molecules of a peculiar structure, we can prove that the latter will exert opposite forces in certain opposite directions; in other words, be endowed with polarity. Imagining another arrangement, we have, as its result, the equal and balanced action which produces liquidity, or the repulsion which characterizes aeriform bodies. Thus the special arrangements of atoms to form molecules, and of these molecules to form masses, may be conceived to give rise to all the various properties of bodies, as hard and soft, tough and brittle, opaque and transparent, sweet and bitter, poisonous and alimentary.

163. The probability of this view of the constitution of matter is strengthened by what is known of the effects of molecular arrangement in chemical phenomena. It has been proved that sugar and starch, and woody fibre, consist of the same chemical elements, united in the same proportions; that the substance carbon has at least three distinct conditions, that of the diamond, of graphite, and of charcoal; that pure chalk and white crystaline marble are identical in composition. These are only a few of the very numerous instances afforded by chemistry of bodies agreeing precisely in composition, and at the same time having different and even opposite properties.

MECHANICS OF SOLID BODIES.

ELEMENTS OF STATICS AND DYNAMICS.

CHAPTER VIII.

COMPOSITION OF MOTION AND MOVING FORCES.

164. A material point or body cannot have more than one absolute motion at the same time, but it may have two or more relative motions. This union of different relative motions in the one motion of the point or body, is what is called the composition of motion.

165. The motions which are thus conceived to be united, are called *Elementary Motions*, and the single motion which includes them all is called the ***Resultant Motion.***

All the phenomena of the composition of motions may be referred to the following proposition: —

166. Composition of two uniform right-line motions. — ***If a material point be impressed at the same time with two uniform motions having the direction and proportion of the sides of a parallelogram, it will have a uniform motion in the diagonal.***

This theorem is in reality little more than a formal expression of the third law of motion. Thus, if we suppose a particle at A (fig. 20) to be impressed with a uniform motion which would carry it to B in one second, and with another uniform motion which would carry it to C in the same time, it will, by the principle of the independence of motions, so move as to maintain both these motions entire.

We may conceive the particle to move uniformly along the line $A\ B$ in one second, while that line is transferred with a uniform motion, in which it preserves its parallelism, to the new position *C D*. The particle thus moving along the moving line, will evidently have the two motions, *A B* and *A C*, combined. At the end of the second the particle will have just reached *D*, the opposite extremity of the diagonal. Assume *m* and *n* midway in *A B* and *A C*, and draw *m p*, *n q*, parallel to these lines. As the motion of the point on the line is uniform, it would be at *m* at the end of the first half second, but the uniform motion of the line carries it to the position *n q* in this half second. Therefore the particle at the end of this time has reached the middle of the line *n q*, that is, *r*, the point of intersection. But this lies in the diagonal of the parallelogram; and in the same way it may be proved that at the end of any other fraction of the time, the moving particle will be in the diagonal.

That the motion in the diagonal is a uniform one, is thus proved. The space *A n* described in a half second, is one half as great as the space *A D* described in a second. So the space $A\ r_1$, described in three-fourths of a second, is three-fourths of *A D*. Thus the spaces described along the diagonal are proportional to the time, and therefore the motion must be uniform.

The velocity in the diagonal is to the velocity in either of the sides as the diagonal to that side.

167. This proposition may be simply illustrated by the following experiments: —

(*a*). An ivory ball suspended by a cord and sliding pulley in front of a black-board, may be moved vertically and horizontally at the same time. The actual line of its motion will be seen to lie in the diagonal *A D* (fig. 21).

(*b*). Two rods of smooth wood, or glass, with a ring loosely embracing both at their intersection, may be so moved, each parallel to itself, as to impress two uniform motions on the ring. When this is done, the ring is seen to travel along the diagonal of the parallelogram (fig. 22).

168. COMPOSITION OF A PROGRESSIVE WITH A ROTARY MOTION. An example of this is presented in the motion of a point situated in the rim of a wheel, which is revolving in the direction of a straight line. The point has the forward motion of the wheel, and has a motion of rotation around

its centre. The direction of the latter is continually changing; sometimes opposing, and sometimes conspiring with the motion of progress. Thus the actual velocity of the point is continually varying, as well as its direction. The form of its path, including at each moment both its motions, is the curve called the cycloid (fig. 23).

169. Composition of two moving forces. — When two or more forces act at the same time upon a material particle or mass, so as to impel it into motion, this motion has a definite direction and velocity. Hence we may conceive of a single force, which, acting in that direction, would impart to the mass the same velocity. This is what is called a ***Resultant Force***, or simply a ***Resultant***, and the forces which unite in producing the motion, are called the ***Elementary Forces***, or the *Components*. Thus, in any given case, the resultant is the efficient force which may be conceived to be generated by the joint action of the elementary forces.

170. The procedure according to which we determine the direction and intensity of the resultant of any given forces acting jointly on a particle or mass, is called the ***composition of forces***.

As this may be most clearly treated by a graphical method, it is necessary to premise a word of explanation.

171. Representation of forces by right lines. — It is obvious that, to define a given force, is to state the direction and magnitude of the pressure or motion which it produces; in other words, to state the direction and magnitude of the force. This may be done most simply by using a right line as a picture or representation of the direction and amount of the force. When two or more forces are to be thus represented, the lines must have the same relative directions, or make the same angle, as the forces and their lengths must be in the same proportion as the intensities of the forces.

Thus, suppose a force valued at 160 to be combined in its action with a second valued at 50, and suppose that they are inclined to one another at an angle of 90°. We draw two indefinite lines at right angles, and assuming any unit of space, suppose 1 inch for every 10 units of force, we mark off on the one line 16 inches, and on the other line 5 inches. The lines thus limited will evidently represent the two forces in their relative directions and intensities. (Fig. 24.)

The fundamental Theorem in relation to the composition of forces, is as follows:

172. *Two moving forces, having the direction and proportion of the sides of a parallelogram, produce a resultant moving force having the direction and proportion of the diagonal.*

This proposition is directly deducible from that of the composition of motion.

Let *A B*, *C D* be the parallelogram of motions before explained, *A B* and *A C* representing two uniform motions, *A* and *A D* their resultant motion. The lines *A B*, *A C*, *A D*, are in the same proportion as the velocities with which they are described. But these velocities are proportional to the moving forces by which they are generated in a given time. Therefore the forces which produce the motions in *A B*, *A C*, and *A D*, are in the proportion of these lines. that is, the two elementary forces and the resultant force bear to each other the proportion of the lines *A B*, *A C*, *A D*. (Fig. 25.)

The parallelogram thus formed upon the lines representing the combined forces, is called the *Parallelogram of Forces.*

Since the diagonal *A D* is by geometry in all cases less than the sum of the adjacent sides, *A B* and *A C*, it follows that the *resultant* of any two oblique forces is always *less than the sum of the forces.* When the angle at *A* vanishes, and the two forces are thus brought to act the *same way*, the resultant becomes equal to their *sum.*

When the angle *A* becomes 180°, that is, when the two forces act in *opposite directions*, the resultant is equal to their *difference.*

173. Examples of the composition of two moving forces, acting at a point.

(*a*). In the impulsion of an arrow by the recoiling movement of a bow, (fig. 26,) two forces are caused jointly to act upon the feather-end of the arrow. For each segment of the string is pulled in the direction of its own length by the expanding bow, and thus a force is applied at *a* in the direction *a B*, and another in the direction *a C*. Denoting the former by *a b*, and the latter by the equal line *a c*, we complete the parallelogram upon these lines. The diagonal *a d* marks the direction of the impelling force, and measures its intensity as compared with the forces applied by the two segments of the string.

(*b*). Suppose a canal boat to be drawn by means of two diverging ropes attached at the bow, the horses pulling steadily on the opposite banks of the canal. Let the tractional force of each, as measured by the spring balance, be 50 pounds, and let the angle of the ropes in the first place be 120 degrees. Assuming 1 inch to represent 5 pounds of force, we draw two lines at an angle of 120°, and lay off adjacent to the angle, a distance of 10 inches on each. These distances will represent the two forces. We then complete the parallelogram, and draw its diagonal. Each inch on this line will correspond to 5 pounds. Hence, multiplying the number of inches in the diagonal by 5, we have, in pounds, the value of the resultant of the tractional forces applied. (Fig. 27.)

When the angle of the ropes is 120°, the diagonal will be equal to either side; and, therefore, the resultant action of the two forces in the direction of the motion, is only equal to the single action of one of them in its own direction.

When the angle is less, the resultant is greater; thus in the right hand figure, where the angle is 90°, the resultant is about nine-tenths of the sum of the two forces.

174. *To find the resultant of any number of forces of known magnitudes and directions, applied to a given material point.*

Let P_1 P_2 P_3 (fig. 28) be the given forces. Assuming a point A, draw the indefinite lines $A\,X$, $A\,Y$, $A\,Z$, at angles corresponding to the known directions of the forces, and upon these lines, according to any uniform scale of measurement, take distances $A\,B$, $A\,C$, $A\,D$ in the same proportion as the forces, so that $A\,B$ may represent P_1, and $A\,C$ and $A\,D$ may represent P_2 and P_3 respectively. Then on $A\,B$ and $A\,C$ construct a parallelogram, and draw its diagonal $A\,R_1$. By the previous proposition this will measure the resultant of P_1 and P_2. Next construct a parallelogram upon $A\,R_1$ and $A\,D$, and draw the diagonal $A\,R_2$. This line will measure the resultant of $A\,R_1$ and $A\,D$. But $A\,R_1$ measures that of P_1 and P_2. Therefore $A\,R_2$ measures the resultant of P_1, P_2, and P_3.

175. The resultant of any number of forces which act

jointly on a material point, may be determined in the same way, by successively compounding the first with the second, and their resultant with the third, and this resultant again with the fourth, and so on to the last force.

It is evident that this procedure is not limited to forces which all lie in one place, but is equally true for forces having any directions, provided they all concur at the same point.

CHAPTER IX.

COMPOSITION OF BALANCED PRESSURES, OR EQUILIBRIUM.

176. WHEN two or more forces are so applied to a material point as to counteract one another and hold the point at rest, they are said to be in *equilibrium.*

If there be only two forces concerned, it is clear that they must be equal and opposite to produce equilibrium. When three forces, P_1 P_2 P_3 are in equilibrium about a point, (fig. 29) each one of them balances the united effect of the other two. But this united effect is equal to what would be produced by the resultant of the other two. *Each force must, therefore, be equal and opposite to the resultant of the other two.*

Thus, P_1 is equal and opposite to the resultant of P_2 and P_3. So again P_2 is equal and opposite to the resultant of P_1 and P_3.

177. *When three forces are in equilibrium around a point, their intensities are proportional to the adjacent sides and diagonal of a parallelogram drawn in their respective directions.* For if P_1 P_2 and P_3 be in equilibrium, and we complete the parallelogram on $A\,B$ and $A\,C$, the measures of P_2 and P_3, the diagonal $A\,D$ must be the measure of P_1, since it is the measure of the resultant of P_2 and P_3.

Instead of using the two sides and diagonal of the parallelogram for measuring the forces, we may use the three sides of one of the triangles forming the parallelogram. Thus $B\,D$, $A\,B$, $A\,D$ are proportional to $P_3\,P_2\,P_1$. This is merely substituting $B\,D$ for its equal $A\,C$. Hence —

178. *When three forces are in equilibrium about a point, they are measured in direction and intensity by the three sides of a triangle drawn parallel to their lines of action.*

This is the principle of the TRIANGLE OF FORCES, and is illustrated experimentally by the apparatus (fig. 30.)

179. It is evident that the triangle thus drawn will measure the two component forces and their resultant, as well in the case of moving forces as of quiescent pressures.

180. ILLUSTRATIONS OF EQUILIBRIUM, AND THE COMPOSITION OF PRESSURES.

Fig. 21. (*a*). A weight sustained at the junction of two beams which rest against abutments, (fig 31.)

The weight compresses the beams, which react severally in the directions *B A* and *C A*. These two forces directed obliquely upwards balance the action of the weight, by generating an upward resultant, equal and opposite to the weight. If *A R* be taken to represent the weight, then *A B* and *A C* or *B R* will represent the reactions of the beams severally. But these reactions are equal to the compressing actions of the weight or the pressures on the abutments. Therefore, *A B* and *A C* will represent the pressures on the corresponding abutments. The sum of the pressures on the two abutments is obviously greater than the whole weight. Since $A\ B + A\ C$ is greater than *A R*.

By enlarging the angle of meeting of the beams, we can make the *pressures on the abutments greater to any extent than the whole weight.* Thus in (fig. 32) *A B* and *A C* are each greater than *A R*.

g. 22. (*b*). A weight sustained at the junction of two beams which are fastened into a wall, (fig. 33).

The horizontal beam *A* is compressed, and reacts outwards or from the wall. The oblique beam *A X* is stretched and reacts upwards or towards *X*. The weight acts vertically downwards. Hence, if *A R* measure the weight, *A C* and *A B* will measure the tensions exerted on the oblique and horizontal beams respectively.

ig. 2 The principle of (*a*) is applied in the *Toggle-joint* (fig. 34.) A given pressure, *A D*, applied at *A* will produce greater pressure along *A b* and *A c* in proportion as the angle at *A* is enlarged.

CHAPTER X.

RESOLUTION OF FORCES.

181. SUPPOSE *A D* (fig. 35) to represent a given force, and let it be required to find two forces, which, by their joint action will produce the same effect as *A D*, and let it be a condition, that these forces are to be parallel to the lines *L* and *M* respectively. It is only necessary to construct on *A D* a parallelogram whose sides shall be parallel to *L* and *M*. These sides, *A B* and *A C*, will measure the forces required.

This procedure is called the resolution of forces, and the force *A D* is said to be resolved into *A B* and *A C*.

182. EXAMPLES OF THE RESOLUTION OF FORCES.

(*a*). A mass *W*, of known weight, say 1000 pounds, is to be raised from the bottom of a ravine by ropes applied in the sloping directions of the surface. What force must be applied to each rope? (Fig. 36.)

Drawing the vertical line through *W* and the inclined lines at their proper angles, we assume a certain length, say ten inches, to represent *W* or 1000 pounds. Completing the parallelogram, the lines *a* and *b* will, on the same scale, give the force in pounds to be applied at the ends of the ropes above, in order just to balance *W*. Any further force will cause its ascent.

(*b*). When an impulse or pressure acts obliquely against a surface, we find the efficient force by resolving the oblique force into two, the one perpendicular and the other parallel to the surface. The perpendicular force being that which causes the approach and generates the repulsion at the surface, is the efficient force. The parallel force is inoperative.

183. This mode of resolving an oblique force is of extensive application in mechanical problems. The following are examples of its use.

1st. When a ball impinges on a plane surface in any oblique direction *M R* (fig. 37), and we wish to determine the force of its impact. We first assume a distance *A R* as the measure of the momentum or entire force impressed upon the ball. This line measures also the force with which the ball would impinge, supposing the direction of its motion to be at right angles to the surface. We now resolve the force $A\ R$ into the perpendicular force $A\ P$, and the horizontal force $A\ Q$. The former is the efficient force, and is alone expended in the impact. The latter remains, and tends to carry the body along the surface.

2d. Suppose a stream of water or air to press continually with a known force, and in a known angle of obliquity. against a given plane $L\ M$ (fig. 38), and that it is required to ascertain the amount of pressure which is exerted at right angles to the surface. We assume a line $A\ R$ at the proper nclination to represent the entire moving force or pressure of the stream, and we resolve it as before into a perpendicular and parallel force. The latter measures the efficient pressure on the surface.

CHAPTER XI.

OF FORCES ACTING AT DIFFERENT POINTS OF A BODY OR CONNECTED SYSTEM.

184. In the preceding propositions, we have conceived the forces to be all applied to the same material point or particle. We come now to consider the action of forces when applied to different points of the same connected mass.

The mechanical principle which forms the basis of our geometrical reasoning in all questions of this kind, is that of the *Transferability of Force.*

185. The simplest case of such actions is, where *a force is applied to one end of a rod or line of particles in the direction of the length, the other end being fixed.* (Fig. 39.)

If the force be one of extension, the line is stretched until the attraction between every pair of adjacent particles becomes equal to the force applied. Thus the particle B at the remote end, is attracted or pulled by x with a force equal to *P*. Hence the force *P* seems to be transferred to *B*, while in fact the extension has excited a force equal to *P* every where along the row of particles.

When the force is one of compression, the particles of the rod will be driven nearer together, until the repulsion thus excited between the adjacent particles is equal to *P*. Thus *B* is repelled by *x* with a force equal to *P*, and hence *P* seems as before to be transferred to *B*. In both cases the effect on the obstacle *W* is the same as if *P* were actually transferred to *B*. It is further evident, that the effect of *P* is the same, at whatever point of *A B* it acts.

186. *If equal forces be oppositely applied to the ends of a rod or rope, they will be in equilibrium.* For, by the principle of transference just proved, either force may be supposed to be applied directly to the other (fig. 40.) If the forces be unequal, the moving force will be the difference of *P* and *Q*.

187. *In a body of any form, the efficiency of an applied force is not altered by transferring its point of application any where along the line of direction of the force.* This important principle is proved experimentally thus. Let *B M* (fig. 41) be an irregular mass, balanced on a pivot at *S*, and free to revolve. Let *P* be a force applied at *A*. Attach a weight *W* at *B* sufficient to counterpoise the action of *P*. If now we transfer *P* successively to *a b c*, &c., preserving its direction unchanged, we shall find that it continues to be in equilibrium with *W*.

188. APPLICATION OF THE PRINCIPLE OF TRANSFERENCE TO THE COMPOSITION OF FORCES, ACTING AT DIFFERENT POINTS OF A MASS OR SYSTEM. — Suppose *P* and *Q* to be applied at *A* and *B*, respectively, in the mass *S M* (fig. 42); by transference we may suppose *P* to be applied at *S*, which is a point in its direction, and the same of *Q*. We therefore construct a parallelogram as if the forces had been applied at *S*, and the diagonal *S V* will give the amount and direction of the resultant of the forces *P* and *Q*, applied at *A* and *B*. Again, this resultant will have the same effect wherever it may be applied in this direction, as at *a* or *b* and *C*.

By an experiment similar to that above cited, it may be shown that when the directions of the forces *P* and *Q*, prolonged, meet outside of the limits of the mass (as in fig. 43) at *S*, the resultant is the same in amount and direction as if the mass extended beyond *S*, and the forces, as in the preceding example, were transferred to that point. Hence here, as in that case, we prolong the directions of the forces until they meet, consider them as applied at the point of meeting, *S*, and completing the parallelogram, take its diagonal for the resultant.

189. If in either of the last cases a pivot be placed in the line of the resultant, within the mass, as at *a* or *b*, the resultant may be considered as applied directly to the pivot. But its pressure against the pivot will, by the law of mutual action, excite an equal and contrary reaction at the point of contact.

190. *Hence, when the pivot on which a mass turns is placed any where in the line of the resultant of the applied forces, the mass will be held in equilibrium ; and conversely, when the applied forces are balanced, their resultant passes through the pivot.*

191. It is evident that the joint action of the forces *P* and *Q* will be balanced by a force equal to the resultant *S V*, provided the force be applied at some point in the line of the resultant, and in the opposite direction. Whence it follows that *if any three oblique forces be applied to different parts of a mass, and hold it at rest, these forces would, if applied to a material point or particle, be in equilibrium with one another.*

We may, therefore, in all such cases, compare the three non-concurring forces as if they met at one point, by constructing the triangle of forces in the same manner as before.

192. It is evident that the three forces thus acting simultaneously on the body *L M* at the points *A B* and *C*, (fig. 44,) are not directly in equilibrium among themselves. The molecular forces exerted by the extensions or compressions they produce, are in equilibrium in their action on each particle, and each external force is in direct equilibrium with the molecular force operating at the point where it is applied.

CHAPTER XII.

OF STATICAL MOMENTS.

193. Measure of the efficiency of Forces acting around a fixed point or pivot. — *The efficiency of a force to rotate a mass around a fixed point, is measured by the product of the force into its perpendicular distance from this point.*

This fundamental principle may be thus proved by experiment: —

Let *S M* (fig. 45) be a mass moveable about *S*, and *P* a force applied any where along *P A*, at the perpendicular distance *S A*. Suppose the weight *W* to balance its action. Now, if *P* be removed to any other position, as *B P*, retaining the same perpendicular distance from *S*, that is $SB = SA$, we will find the same weight *W* to hold it in equilibrium. If now it be applied at half the distance, as *C P*, $\frac{1}{2}$ *W* will balance it; if at double the distance, then 2 *W* will be required to balance it, &c.

Again, if at a double distance we apply a force equal to 2 *P*, then 4 *W* will be required for equilibrium.

Hence, the efficiency of the force is proportional to the force multiplied by the distance.

This product is what is called a Moment.

194. *If the two forces, $P_1 P_2$, act on the same side of S, that is, in the same direction of rotation, the moment of their resultant is equal to the sum of the moments of elementary forces;* for the resultant being substituted for the two forces, will have equal efficiency. But the efficiency of the resultant is measured by its moment, and the joint effect of P_1 and P_2 is measured by the sum of their moments. Therefore moment of resultant = sum of moments of P_1 and P_2 (fig. 46.)

195. *If P_1 and P_2 act on opposite sides of S, that is, in opposite directions of rotation, the moment of their resultant is equal to the difference of moments of elementary forces.* For the efficiency of the resultant = joint efficiency of

P_1 and P_2. But the efficiency of the resultant is measured by the moment of the resultant; and the joint efficiency of P_1 and P_2 is measured by the difference of their moments, because their rotary action is opposite. Therefore, the moment of the resultant = difference of moments of P_1 and P_2 (fig. 47.)

196. When the two forces, P_1 and P_2 are in equilibrium, the resultant passes through the pivot, (190) and the moment of P_1 = moment of P_2, because their rotary effects are equal. The same principle may be extended to any number of forces in equilibrium, around a fixed point or axis.

In this case, *the sum of the moments tending to turn the mass in one direction, is equal to the sum of the moments tending to turn it in the opposite direction.* In the (fig. 48) moment of P_1 + moment of P_2 = moment of P_3 + moment of P_4.

This is the great principle of the equilibrium of forces around an axis.

197. In the above examples, the applied forces, P_1 P_2, &c., are not in equilibrium among themselves. The action at the pivot, which is equal and opposite to their resultant, balances their combined action, and thus the forces really in equilibrium are P_1 P_2, &c., and the reaction at the pivot.

CHAPTER XIII.

OF PARALLEL FORCES.

198. Suppose two forces, P_1 P_2, parallel and acting the same way, to be applied to the points A and B of the solid. It is required to find the position and magnitude of their resultant. (Fig. 49.)

1st. *Position of the Resultant.* — The resultant must be parallel to the forces P_1 P_2, because they are parallel to one another. It must, moreover, intersect the line A B in a point which, if assumed as a pivot, would cause equilibrium. Suppose C to be the point in question. Draw Cx and Cy

perpendicular to the forces. Since there is equilibrium around C, we have the moment of P_1 = moment of P_2, that is, $P_1 \times Cx = P_2 \times Cy$. But $AC : BC = Cx : Cy$. Therefore $P_1 \times AC = P_2 \times BC$. Hence the resultant *divides A B inversely as the forces.*

2d. *Magnitude of the Resultant.* — As P_1 and P_2 act the same way, the resultant = the sum of P_1 and P_2, or $R = P_1 + P_2$. That is, *when parallel forces act the same way, their resultant is equal to their sum.*

Example: Suppose $P_1 = 10$ pounds, $P_2 = 20$ pounds, and $AB = 40$ inches, to find the place of the resultant. Divide AB into $10 + 20 = 30$ parts. Lay off 30 of these equal parts next to P_1, and 10 next to P_2. The point of division will be the place of the resultant. Amount of resultant $= 20 + 10 = 30$ pounds.

199. When any number of parallel forces acting on a mass, are in equilibrium around a pivot, the sum of moments tending in one direction, is equal to sum of moments tending in the opposite direction.

200. *To find the position and magnitude of the resultant of three or more parallel forces*, as $P_1\ P_2\ P_3$ (fig. 50.) First find the place of the resultant of P_1 and P_2. Suppose this to be at D. This force will be equal to P_2 and P_3. Join D and C. Consider P_1 and P_2 as a single force acting at D, and find the place of the resultant of this and P_3 by dividing DC in the inverse ratio of the two forces at its extremities. Let G be the point thus found. The resultant of $P_1\ P\ P_3$ acts through G, and is parallel to the three forces, and equal to their sum.

201. The point of application of the resultant of parallel forces is the same, whatever be the direction of the forces in reference to the mass. For as in (fig. 49) G must always divide the line CD, joining the points of application of the partial resultant and the last force in a fixed ratio.

CHAPTER XIV.

OF THE CENTRE OF GRAVITY.

202. Every ponderable mass may be regarded as made up of integrant particles, all of which gravitate equally and in parallel lines. Thus a body of whatever form is in effect a group of innumerable equal parlalel forces of gravity. These forces must have a resultant equal to their sum, that is, to the whole weight of the mass, and by (201) the point of application of this resultant must remain the same, whatever the direction of the forces in regard to the mass; in other words, whatever the attitude of the mass itself. This point is called the CENTRE OF GRAVITY.

We may, therefore, define *the centre of gravity to be the point through which passes the resultant of the gravities of all the particles of a body, in every position of the mass.*

In considering the action of a mass as a whole, in virtue of its gravity, we may properly regard its weight as all concentrated, in the shape of a single force, at the centre of gravity, instead of being distributed over the mass.

203. It follows, from this, that to find the centre of gravity of a mass or system, is merely to find the place of the resultant of the parallel forces of gravity acting upon its parts.

204. *To find the centre of gravity of a physical line.* 1st. In the case of a straight line, (fig. 51,) the resultant of the two extreme forces is at the centre; that of the two next at the centre also; and so on, until all the particles of the line are included. Therefore the centre of gravity is at the middle of a right line.

2d. When two right lines are united at an angle, (fig. 52,) the centre of gravity of $A\ B$ is at the middle C, and that of $B\ C$ at the middle C. The whole weight of $A\ B$ and of $B\ C$ respectively may be transferred to C and C. Join those points, and take a point, G, which divides $C\ C$ inversely as the weights of $A\ B$ and $B\ C$.

The centre of gravity of a curved line (fig. 53) lies somewhere within the concavity as in G.

205. *To find the centre of gravity of a physical surface.* 1st. The centre of gravity of a circle, square or other symmetrical plane surface is at the centre of its form. So the centre of gravity of the surface of a sphere or cube or other symmetrical envelope is at the centre of form.

2d. To find the centre of gravity of a *plane triangle*, from an angle C (fig. 54) draw a line bisecting the opposite side. As this bisects all the lines parallel to that side, it passes (204) through the centre of gravity of all these lines. It must therefore contain somewhere their common centre of gravity, that is, the centre of gravity of the triangle. From any other angle as B, draw a line to bisect the opposite side A C. This also will contain the centre of gravity of the triangle. Therefore the centre of gravity of the figure must be at the intersection G of these bisecting lines.

3d. To find the centre of gravity of a *polygon*, (fig. 55.) If the figure have four sides, divide it into two triangles by the diagonal A C. Find the centre of gravity of each triangle. Suppose g and g to be these centres, join them by the line g g. Then, we may suppose the weight of A B C concentrated at g, and that of A C D at g. Divide this line at G, in the inverse ratio of these weights, and G is the centre of gravity of A B C D.

206. *To find the centre of gravity of a solid.* 1st In the sphere, cube, and other symmetrical solids, the centre of gravity coincides with the centre of the figure.

2d. To find the centre of gravity of a *triangular pyramid*, (fig. 56.) First find the centre of gravity of the base, suppose at g, and join this point with the vertex V. Second, find the centre of gravity of one of the sides as A C V; let this be at g_1. Where V g and B g_1 intersect, will be the centre of gravity of the solid. For the pyramid is made up of triangles parallel and similar to the base A B C, placed one above another, continually diminishing in surface. The line V g will pierce them all at their centres of gravity. Therefore the weight of the pyramid may be regarded as distributed along that line. Hence the centre of gravity of the whole figure must be in the same line. In the same way it may be proved that the centre of gravity is situated somewhere in $B\,g_1$. Therefore it is at G. The point G divides the line $V\,g$ in the proportion of 1 to 3, and therefore $A\,g = \frac{1}{4}\,V\,g$. (Fig. 57).

207. *To find the centre of gravity of systems of connected masses.* — 1st. To find the centre of gravity of two bodies, connected by a rod neglecting the weight of the rod. Find the centre of gravity of the masses L and M, suppose at A and B. Join $A\,B$. The whole weight of L may be considered as acting at A, and that of M at B. We now find the place of their resultant by dividing $A\,B$ at G in the inverse ratio of the weights, or making $L \times A\,G = M \times B\,C$. G is the centre of gravity required, (fig. 58).

2d. In the case of three or more bodies neglecting the weight of the connecting matter. Find the centre of gravity of L and M as before, suppose at G. The sum of the weights L and M may be regarded as acting at G. Join this point with C, the centre of gravity of the third body N, and divide the line $G\,C$ in the inverse ratio of $L + M$ to N, or so that $L + M \times G\,x = N \times C\,x$. The point x is the centre of gravity of the three masses, (fig. 59).

3d. To find the centre of gravity of two or more bodies, and their connecting rod. Find G the centre of gravity of L and M, and C that of rod. Join $G\,C$, and divide it at x in the inverse ratio of $A + B$ to $D\,K$, (fig. 60).

208. OF EQUILIBRIUM AS DETERMINED BY THE POSITION OF CENTRE OF GRAVITY. — First. In the case of *suspended bodies*, equilibrium occurs under either of the following conditions.

1st. When the centre of gravity is vertically under the point of support.

2d. When it is vertically above the point of support.

3d. When it is at that point.

In all these cases, the resultant weight acting through the centre of gravity is directly antagonized by the resistance at the point of support. The equilibrium is of three different kinds, viz. *stable*, *tottering*, and *neutral*.

(*a.*) *Stable Equilibrium.* When the centre of gravity is under the point of support, the body, on being disturbed from its vertical position, will return to it. This is what is called stable equilibrium, and is illustrated by (fig. 61.)

Let *G* be the position to which, by moving the body, the centre of gravity has been moved. The weight acting through *G*, vertically downwards, and the supporting force in the direction *G S*, the motion of *G* must be downwards in the curve, so as to restore *G* to the vertical.

(*b*). *Tottering Equilibrium.* — When the centre of gravity is above the point of suspension, the body, on being moved from its equilibrium, will revolve until it is overturned. This is tottering equilibrium.

Let *G* be the place of the centre of gravity, when the body is disturbed. The whole weight acting downwards through *G*, and the supporting force in the direction from *S* towards *G*, the point *G* will move in the descending curve until it takes a position, G_1 vertically under *S*. (Fig. 62.)

(*c*). *Neutral Equilibrium.* — When the centre of gravity coincides with the point of suspension, the body will remain at rest in any attitude. For in this case the weight and supporting force are both applied at the point *G*.

209. *The force of stability of a suspended body* is measured by the amount of weight or other force necessary to retain it out of the vertical at a certain angle of deflection, and is equal to the force with which it tends to recover its verticality.

210. Measure of the Stability of a Suspended Body. — The force of stability is greater in proportion to the weight of the mass, and in proportion to the distance of the centre of gravity below the point of suspension.

Let a body suspended by *S* (fig. 63,) be forced into an oblique position by the attached weight *P*. If the centre of gravity be at *G*, then the weight of the body will act through *G*, and the efficiency or moment of this force will be equal to the mass multiplied by $Sx = Mass \times Sx$. But if the centre of gravity be at *G*, the moment of the weight will be $= Mass \times Sy$. Hence, in order to retain the body in the same oblique position, the force *P* must be increased in the proportion of Sx to Sy; that is, in the proportion of SG to SG_1.

Suppose now that, while the centre of gravity is thus removed to a greater distance below *S*, the weight or mass itself is increased. Then since the force P, in order to retain the mass in the same position as before, must balance the moment of the weight, it must be increased as the mass is increased. Thus the force P must vary in the same proportion as *Mass* × SG.

Hence, the ***stability of a suspended body is measured by the product of the mass of the body into the distance of its centre of gravity, from its point of suspension.***

211. OF THE EQUILIBRIUM OF BODIES RESTING ON A HORIZONTAL PLANE. — General Law. ***Equilibrium occurs whenever the vertical, passing through the centre of gravity, falls within the limits of the base.*** For in every such case the resistance of the base will directly oppose the weight.

(*a*). In the case of a body bounded by flat surfaces, there will be stable equilibrium when the body rests upon a side, and tottering when upon an angle. In the former case, the force through the centre of gravity tends to restore the mass to its former position; in the latter, to overturn it. As the body is rolled over, the stable and tottering conditions of equilibrium succeed each other alternately. (Fig. 64.)

(*b*). In the case of a body having a curved surface, the equilibrium is determined by drawing a line at right angles to the surface, and marking where it intersects the vertical axis. This point is sometimes called the ***metacentre.*** The position of the metacentre in regard to the centre of gravity determines the kind of equilibrium.

212. Thus (in fig. 65,) the line PM being drawn at right angles to the surface at the point of support, intersects the axis or line which symmetrically divides the mass, at the point M, which is, therefore, the metacentre.

The supporting force, which is nothing more than the molecular repulsion between the surfaces at P, acts in the direction of PM; that is, vertically upwards and through the metacentre. The weight of the body acts vertically downwards through the centre of gravity. When this latter point is at G, or below the metacentre, the rotary effect of the two forces tends to restore the mass to its vertical position. When the centre of gravity is at G_1, above the metacentre, these forces conspire to overthrow the mass. Hence the following rule.

When the centre of gravity is below the metacentre, the equilibrium is stable; when above, it is tottering.

213. MEASURE OF THE STABILITY OF BODIES RESTING ON A BASE. — Two kinds of measure are adopted, viz., the statical and dynamical.

First. *Statical Measure of Stability.* — This depends on the amount of force necessary to begin rotation from the position of stability (fig. 66). A force being applied just sufficient to cause the beginning of rotation around *A*, this will be balanced by the weight of the mass acting on the line *G C*. The moment of this weight tending to restore the body, is = *Weight* × *A C*. Hence the rule, that the statical stability is proportional, jointly, to the weight of the mass, and the distance of the vertical from the angle around which the mass moves. It is obvious that this stability does not depend on the vertical height of the centre of gravity, but upon the breadth of the base. In fig. 66, the stability is the same for both positions of *G*.

Second. *Dynamic Measure of Stability.* — This depends on the amount of moving force expended in raising the centre of gravity from its lowest to its highest position. Thus, (in fig. 67,) when the body is compelled to revolve around *A*, the centre of gravity ascends from *G* to *H*. The line *L H* is its vertical motion. The force expended must evidently be proportional to the weight raised, and to the height through which it is raised. Hence the stability is measured by the weight multiplied by the rise of the centre of gravity; that is, *Weight* × *L H*. Here we see that the higher the centre of gravity is above the base, the less is the value of *L H;* and, therefore, the less the stability.

214. OF THE EQUILIBRIUM OF BODIES RESTING ON AN INCLINED PLANE. — *General Law. When a body with a flat base rests on an inclined plane, there is equilibrium when the vertical line, drawn through the centre of gravity, falls within the lower angle of the base.* As the height of the centre of gravity increases, the intersection of this vertical approaches the lower angle. The statical measure of the stability is in this case the weight multiplied by the distance of the vertical from the lower angle of the mass. Hence the stability diminishes, in proportion as the centre of gravity is higher. (Fig. 68.)

215. GENERAL CRITERION OF EQUILIBRIUM. — In every case,

whether of suspension or support on a base, the motion of the centre of gravity, when the mass is disturbed, furnishes a criterion of the kind of equilibrium.

1st. When the centre of gravity moves in an ascending curve, the equilibrium is stable.

2d. When its motion is descending, the equilibrium is tottering.

3d. When it either does not move at all, or moves horizontally, the equilibrium is neutral.

Thus in the ellipse (fig. 69,) we have stable equilibrium in *a*, tottering in *b*, and neutral in *c*.

216. In the case of a body with a flat base resting on a sphere, there is stable equilibrium when the distance of the centre of gravity from the surface is *less* than the radius. Tottering, when *greater;* and neutral, when the two distances are *equal.* (Fig. 70.)

In the first case, the centre of gravity *ascends* when the body is disturbed; in the second it *descends*, and in the third it *moves horizontally.*

217. *Stability of the attitudes of animals.*—Walking, running, &c., may be described as a repeated falling and restoration of the body. In walking, when the joints are flexible, the centre of gravity of the body describes a gently undulating curve. But when the knees are stiff, as with wooden legs, the line of its motion is made up of curves which are *suddenly* changed at each step. (Fig. 71.)

218. So when *a carriage without springs* passes over a rough pavement, the centre of gravity of the whole mass rises and falls suddenly with the irregularities of the surface, and thus the path of this point is made up of a series of short lines in a zigzag form. But if the carriage be mounted *on springs*, the centre of gravity will change the direction of its motion in a gradual manner, thus rounding off the angles of the path described in the preceding case, and changing it into an undulating line.

219. In the erect position of the human form, when the heels are placed in contact (fig. 72), the greatest stability is secured by giving the feet an inclination of ninety degrees. It is evident that when the feet are placed together, and parallel, they will form a smaller base than when thus

inclined; and the same will be true if they are made to diverge very much. The quadrangles $A\ B\ C_1\ D_1$ and $A\ B\ C_2\ D_2$ represent the areas of support in these two cases, while $A\ B\ C\ D$ is the area when the inclination of the feet is a right angle. By geometry it may be proved that $A\ B\ C\ D$ is greater than the area formed at any other inclination.

If the heels be placed at a distance apart equal to the length of the foot, as in (fig. 73), it may by like reasoning be shown that the position of maximum stability is where the feet are inclined at an angle of sixty degrees.

220. Of the centre of momentum. — This is the point between two attracting masses, at which, in virtue of their mutual attraction, they would meet.

Thus, let L and M (fig. 74) be two unconnected bodies attracting one another. They will approach, and at last meet at some point, C, which is their centre of momentum. This point divides the distance $A\ B$ in the inverse ratio of the masses, making $L \times A\ C = M \times B\ C$. For by the law of mutual actions, the momentum of $L =$ momentum of M. Hence at each moment of their approach, the velocity of L is to the velocity of M as M is to L. Therefore the spaces described are in the same ratio; that is, $A\ C : B\ C = M : L$, and $L \times A\ C = M \times B\ C$.

Thus the position of the centre of momentum is the same which the centre of gravity would have, if the bodies were connected together, and were urged by their gravity towards some third mass.

If E and M (fig. 75) represent the earth and moon, the centre of momentum, C, will divide the line $E\ M$ in the inverse ratio of the masses. But if the earth and moon were connected by a bar, and gravitated in parallel lines towards the sun, then C would be the centre of gravity of the earth and moon. From this coincidence of the centre of gravity with that of momentum, the term centre of gravity is commonly used for both, although the mechanical considerations to which these names refer are entirely distinct.

221. *To find the distance of the centre of gravity, or momentum of two or more bodies from a given plane.*

Let A and B be two bodies, and G their centre of gravity.

Draw $A\,x$, $B\,y$ and $G\,z$ perpendicular to the plane $H\,R$; also $m\,G\,n$ parallel to $H\,R$, and intersecting $A\,x$ and $B\,y$ produced at m and n. Because G is the centre of gravity of A and B, we have $A \times A\,G = B \times B\,G$. But $A\,m : B\,n = A\,G : B\,G$. Therefore $A \times A\,m = B \times B\,n$. Substituting $A\,x - G\,z$ for $A\,m$ and $G\,z - B\,y$ for $B\,n$, we have $A \times A\,x - A \times A\,z = B \times G\,z - B\,x\,B\,y$. This, by transposition, gives $A \times A\,x + B \times B\,y = (A + B)\,G\,z$. And, therefore, $G\,z = \dfrac{A \times A\,x + B \times B\,y.}{A + B}$ (Fig. 76.)

That is, the distance of the common centre of gravity of the masses from the plane, is found by multiplying each mass into the distance of its centre of gravity from the plane, and dividing the sum of the products by the sum of the masses.

When A and B are equal, then $G\,z$ = half sum of $A\,x$ and $B\,y$. That is, the centre of gravity is in the *centre of position*, or has a distance equal to the average distance. Thus, in the surface $L\,M\,N$ the distance of the centre of gravity from $x\,y$ is the average of all the distances of the parts. $G\,A$ is an average of the lines $m\,a$, $n\,b$, &c. (Fig. 77.)

222. Applying this principle to a line, $A\,B$, or surface, $P\,Q$, we see that in these cases the distance $G\,x$ of the centre of gravity of the line or surface from a given axis, $L\,M$ (figs. 78 and 79) is the average of the distances of all the parts of $A\,B$ or $P\,Q$ from this axis. If we suppose $A\,B$ and $P\,Q$ to be carried round the axis, so that $A\,B$ shall generate the surface $A\,B\,C$, and $P\,Q$ shall generate the solid ring $P\,Q\,R\,S$, the different points in $A\,B$ and $P\,Q$ will describe circular paths of different lengths, in proportion to their distances from the axis. But the centre of gravity, G, will in both cases describe a circle having a length which is exactly the average of all the others.

Now the curved surface $A\,B\,C$ is made up of innumerable narrow zones, described severally by the parts m_1, m_2, &c. which make up $A\,B$. But the sum of all these zones, varying from the greatest to the least, is evidently the same as their sum would be if they all possessed the average circumference; that is, the area $A\,B\,C$ is the same as if all its parts were carried around $L\,M$, in a circle equal to that which is described by *C*. Hence it follows that the area of the surface *A B C* is equal to the line *A B* multiplied by the circle described by the centre of gravity of *A B*.

In the same manner it may be shown that the solidity or volume of the ring *P Q R S* is equal to the product of the area *P Q* into the circumference described by its centre of gravity. Hence the following theorems: —

1st. The area generated by the rotation of a line around a fixed axis is equal to the product of this line by the circumference described by its centre of gravity.

2d. The volume or ring generated by the rotation of a plane surface around a fixed axis, is equal to the product of this surface by the circumference described by the centre of gravity.

Example: A circular arc revolves so as to describe a dome-shaped surface. The arc is 30 feet long, and the orbit of its centre of gravity is 200 feet. The area of the dome will be $30 \times 200 = 6000$ square feet.

223. Of the motion of the centre of gravity of a system of bodies. The general principle which governs the motion of this point is as follows: If forces P_1 P_2 and P_3 be applied to the parts *A B C* of any system of unconnected bodies, the motion of the common centre of gravity, *G*, will be the same as if all the bodies had been collected at that point, and all the forces had been applied to the aggregate mass in that position (fig. 80).

CHAPTER XV.

OF THE MECHANIC POWERS.

224. The Mechanic Powers. — These are certain arrangements of solid matter serving as media for the application of forces, and are the elements of which all machines and engines are constructed. They may be classed under the following heads: the Lever, Wheel and Axle, Pulley,

INCLINED PLANE, SCREW and WEDGE. In principle they are reducible to three, viz., the LEVER, the PULLEY, and the INCLINED PLANE.

225. OF THE LEVER. — A lever is any bar or mass, either straight or crooked, resting on a fixed point or pivot around which it may revolve. Such point is called the FULCRUM.

The applied force is called the POWER, and the force which it is designed to balance or overcome, the WEIGHT or resistance.

226. When there is equilibrium through the medium of the lever, the relation of power to weights is as follows:

Moment of Power = Moment of Weight.

This is but a case of the general principle of moments already proved (196). In (figs. 81 and 82,) P and W represent the power and weight respectively, and F the fulcrum. We, therefore, have the equation $P \times F\,M = W \times F\,N$.

As the efficiency of P and W is proportional to the length of $F\,M$ and $F\,N$, respectively, and not to the length of $A\,F$ and $B\,F$, we may designate $F\,M$ as the *power arm* and $F\,N$ as the *weight arm.*

227. Thus the equation of equilibrium of the lever is as follows:

Power × Power arm = Weight × Weight arm.

In this case we assume that the weight of the lever itself is previously balanced, or that the lever is so constructed as to be self-poised on its fulcrum.

228. When there is equilibrium through the medium of a lever, the resultant of the power and weight must pass through the fulcrum. The reaction of the fulcrum is equal and opposite to this resultant, and is thus in equilibrium with the joint action of the power and weight. Each of the three forces, power, weight, and reaction, is equal and opposite to the resultant of the other two.

229. When the lever is a straight bar, and P and W act perpendicularly to its length as in (fig. 83,) the power arm and weight arm coincide with the segments $F\,A$, $F\,B$. In this case *Power* $\times\ A\,F =$ *Weight* $\times\ B\,F$.

Even if P and W are not perpendicular to $A\,B$, provided they are parallel to one another, the relation above given will obtain. Thus in (fig. 84) we have, by moments, $P + F M = W \times F N$. But since $F A$ exceeds $F M$ in the same ratio as $F B$ exceeds $F N$, we may substitute $F A$ and $F B$ in the above equation for $F M$ and $F N$, and thus we have $P \times F A = W \times F B$.

230. For convenience of description levers are arranged under three heads, according to the relative position of the power, weight, and fulcrum. These are called orders of lever.

In a lever of the *first* order, the *fulcrum is between the weight and power.*

In the *second* order, the *weight* is between the *fulcrum and power.*

In the *third* order, the *power* is between the *weight and fulcrum.*

231. *Pressure at the fulcrum.* — When the power and weight are parallel as in the preceding cases, and of known amount, it is easy to determine the pressure on the fulcrum, which is nothing more than their resultant. In a lever of the first order, pressure on fulcrum $= W + P$. Here P and W both act the same way. In the second order, pressure on fulcrum $= W - P$. In the third order, pressure on fulcrum $= P - W$. In the two latter cases P and W act contrary ways.

As in practice, the friction at a pivot or fulcrum is proportional to the pressure, the second order is to be preferred to the first, as involving less friction with a given power and weight.

232. In the second or third order, where the power and weight are applied at right angles to the lever, the moment of $P = P \times A\ F$ and moment of $W = W \times B\,F$. As these moments act in opposite directions of rotation, they must be equal, in order to cause equilibrium. Therefore in the *second and third orders*, as in the first, the equation of equilibrium is *Power* × *Power arm* = *Weight* × *Weight arm.*

233. The distance $A\,F$ of the power from the fulcrum is often called the *leverage* of the power. The efficiency of P augments in proportion to the increase of $A\ F$. The same is true of the efficiency of W as affected by $B\,F$.

234. Familiar Examples of Levers. — ***First Order.*** A crowbar used as in (fig. 85). A poker used by resting it against a bar of the grate. A pair of scissors or pincers, the rivet being the fulcrum. A common steelyard for weighing.

Second Order. The common straw-cutter. A crowbar used as in (fig. 86). An oar used as in rowing (fig. 87). A gate moved on its hinges. The old cider press, (fig. 88). Bar of safety-valve, (fig. 89).

Third Order. A pair of shears. Tongs for grasping a fagot, &c. This form of lever is found very commonly in the mechanism of animals. Thus in the forearm, (fig. 90), the fulcrum being the elbow, the muscle that supports the arm and hand applies its power at A, between the fulcrum and weight. So in the action of the lower jaw, (fig. 91), the fulcrum being at the point f, the muscle that raises the jaw applies its power at A.

235. Of the Bent Lever. — In (fig. 92) we have $P \times F M = W \times F N$. A hammer used to draw a nail is an an example. The power acts in the line $A P$, the resistance in $N B$, here $P \times F P =$ Resistance $\times F N$.

In the packing press for flour, (fig. 93) the power acts in line $A P$, the resistance in $x N B$, and we have $P \times F P$ = resistance $\times F N$. As $A F$ descends, the power arm $F P$ increases, and the weight arm $F N$ grows less. Hence, for a double reason, the efficiency of P is augmented.

236. Of Combined Levers. — In the system of Levers, fig. 94,) let $p_1 p_2 p_3$ represent the power arms, and $w_1 w_2 w_3$ the weight arms of the component levers.

As there is equilibrium in the system, it is obvious that the upward pressure at B proper to the lever $A B$, is balanced by the downward pressure at the same point proper to the lever $B C$; and again, that the downward pressure at C proper to the lever $B C$, is balanced by the upward pressure at the same point proper to $C D$.

Applying the equation of the single lever to each, we have

Pressure at $B \times w_1 = P \times p_1$.

Whence pressure at $B = P \times \frac{p_1}{w_1}$.

But pressure at $C \times w_2 =$ pressure at $B \times p_2$.

Therefore pressure at $C =$ pressure at $B \times \frac{p_2}{w_2}$.

Substituting in this the value of pressure at B, above found, we have

Pressure at $C = P \times \frac{p_1}{w_1} \times \frac{p_2}{w_2}$.

Proceeding in the same manner, for the lever CD, we have

Pressure at $D \times w_3 =$ Pressure at $C \times p_3$.

Whence pressure at D or Weight $W =$ Pressure at $C \times \frac{p_3}{w_3}$.

That is, $W = P \times \frac{p_1}{w_1} \times \frac{p_2}{w_2} \times \frac{p_3}{w_3}$.

Therefore $W \times w_1 \times w_2 \times w_3 = P \times p_1 \times p_2 \times p_3$.

Hence the Equation of Equilibrium of the system of Levers, is as follows:

The power multiplied by the continued product of the power arms, is equal to the weight multiplied by the continued product of the weight arms.

237. Effect of the Weight of the Lever. — In the preceding propositions, the lever has been regarded as self-balanced; or, in other words, as having its centre of gravity in the vertical line passing through the fulcrum. When, however, the centre of gravity is on one side of the fulcrum, the weight of the lever becomes one of the forces tending to turn it round the fulcrum. Thus (in fig. 95,) the weight of the lever acting at G, its centre of gravity, aids the power P to support W. By the principle of moments, therefore, we have the moment of P plus the moment of the weight of the lever referred to G, equal to the moment of W. That is,

$$P \times A\,C + \text{weight of lever} \times G\,C = W \times B\,C.$$

238. Of the Steelyard and its graduation. — In this instrument (fig. 96) the fulcrum is at F, the body to be weighed is placed in the scale or attached to the hook, and its weight is measured by sliding the counterpoise P along the graduated bar. The value of the graduations is ascertained in the following manner:

First. When the steelyard, with the attached scale, is self-balanced. In this case we place the unit of weight, say 1 lb., in the scale, and shift the counterpoise, *P*, until it causes equilibrium. We now mark the point of suspension of *P*. Its distance from $F = FA$, will correspond to 1 lb. of weight. As by moving *P* to distances successively equal to $2\,FA$, $3\,FA$, &c., it will balance 2 lbs., 3 lbs., &c., we graduate *F D* by setting off successive spaces, each equal to *F A*, and marking these divisions 1, 2, 3, 4, &c. In this case, the graduation begins at *F*, the fulcrum.

Second. When the centre of gravity of the instrument, including the scale, is on one side of the fulcrum, as in *G*, (fig. 97). The simplest plan is to suspend from *D* a hollow ball, so loaded with shot as to bring the apparatus into equilibrium; we may then proceed with the graduation as in the preceding case. If, however, we are to use the moveable counterpoise *P* without previously adjusting the instrument to equilibrium, we proceed as follows. Slide the counterpoise until it causes equilibrium. Suppose this to be when it is at *A*. Then load the scale with the unit weight 1 lb., and slide the counterpoise towards *D* until it restores equilibrium. Let its new position be indicated by *a*. The distance *A a* on the steelyard corresponds to 1 lb., a double and triple distance will correspond to 2 lbs., 3 lbs., and so on. Thus the graduations proceed by equal divisions, beginning at the point *A*, where the counterpoise balances the unloaded instrument.

239. Of the Platform Balance, or Weighing Machine.— This consists essentially of three levers below the platform, and the steelyard above, on which the counterpoise is hung. Those beneath are of the second order.

The weight placed on the platform is equally divided (fig. 98) between the levers CB and CB_1, one half resting at A, the other at A_1. The ends of these levers, united by a loosely inclosed ring, are pressed down, and by means of a hook connecting them with the lever *D F*, they pull down this lever by a force applied at *E*. This downward pressure acting through the rod *F G*, on the short arm of the steelyard, is balanced by a proper adjustment of the counpoise *P*.

To compute the ratio of the power to the weight, observe that AC, DE, and GH are weight arms, and BC, DF and HP are power arms. Hence (236)

$$P \times HP \times DF \times BC = W \times GH \times DE \times AC.$$

For example,

Let BC or $BC_1 = 20$ in. AC or $A_1 C_1 = 2$. $FD = 30$. $ED = 10$. $PH = 8$. $GH = 1$. Then $P \times 30 \times 8 = W \times 2 \times 10 \times 1$; that is,

$$P \times 1600 = W \times 20 \text{ or } 80\,P = W.$$

240. Of the Danish Balance. — In this the counterpoise is a load permanently attached to one end of the bar, (fig. 99,) the scale being suspended from the other. The fulcrum is a moveable ring, through which the bar slides. Let $W =$ weight of bar and scale, and let G be the centre of gravity of the whole. Then the constant weight of the instrument acting through G, is the force which is made to counterpoise the weight in the scale.

We therefore have for the equation of equilibrium,

$$W \times Gf = \textit{Weight} \times Af.$$

241. Of the Balance of Equal Arms. — The conditions which regulate the sensibility of this instrument, are thus investigated (fig. 100):

Let A and B be the points of suspension of the scale pans, and let AB be called the line of supports. Let F be the fulcrum, and G the centre of gravity of the beam. In a good instrument the line FG is at right angles to AB, and the point C is midway between A and B. Suppose these conditions fulfilled. Let the scales be loaded equally, each with the weight P. The resultant of these two weights $= 2\,P$, has its point of application at C; we may, therefore, in our reasoning, assume a force $= 2\,P$ acting vertically at C. The weight of the beam acts vertically through G. Suppose, now, a small additional weight w to be placed in the scale which hangs from A. This will incline the beam until the forces through G and C counterbalance it; we have thus, from the principle of moments,

Moment of $w =$ moment of $2\,P$ at $C +$ moment of weight of beam at G.

242. We may now deduce the following inferences:

1st. The greater the weight of the beam, acting through G, the less the effect of *w* in deflecting it, and therefore the less the sensibility of balance.

2d. The greater the distance of *G* below the fulcrum, *F*, the less the effect of *w*, and therefore the less the sensibility.

3d. The greater the force 2 *P* at *C*, that is, the greater the load in the scales, the less the effect of *w*, and therefore the less the sensibility.

4th. The greater the distance of *C*, the centre of the line of supports below the fulcrum, the less the effect of *w*, and therefore the less the sensibility.

5th. The greater the length of *A C*, that is, the greater the length of the beam, the greater the effect of *w*, and therefore the greater the sensibility.

243. It follows, from the above, that the *greater the sensibility of a balance, the slower its vibration;* for the vibration is the effect of the forces through *G* and *C*, tending to restore the beam to its horizontal position, and these forces act with least efficacy, and therefore cause the slowest motion when the leverage is least, that is, when the sensibility is greatest. It is, therefore, incident to the process of weighing, that it is slow in proportion to its accuracy, and that with a very sensitive beam it takes a long time. Hence the importance of being able to adjust a balance to different degrees of sensibility, according to the kind of weighing in view.

244. This is done by raising or lowering the centre of gravity, by the motion of a metallic ball, screwed upwards or downwards, on a vertical wire attached to the middle of the lower edge of the beam (fig. 101). The screw-thread of the wire being very small, a single revolution of the ball will move it up or down a very small distance, and therefore shift the place of the centre of gravity, by an extremely minute amount. When we wish to weigh with great nicety, we adjust the beam by moving the ball upwards; when we wish to weigh quickly, we move it downwards.

245. To reduce the friction at the fulcrum, the contact is made by a triangular prism of hardened steel, or of agate, firmly fastened in the beam, and resting by its lower edge on a polished plane of the same material (fig. 102). A similar arrangement is used for supporting the scales, but the prism is here fixed to the end of the beam with its edge upwards,

and the plane of steel or agate attached to the cords of the scale-pan, by a strong bent wire, rests on the edge of the prism. Supports of this kind are called *knife edges.* It is obvious that for accurate adjustment all the knife edges should be horizontal.

246. To secure greater precision in the use of the balance, the index is prolonged downwards some distance, so as to show the slightest motion of the beam by a graduated scale, in front of which the index vibrates. Sometimes the index projects horizontally from one end of the beam, and moves in front of a vertical graduated scale. In either case, in order to be sure of equilibrium, we need not wait until by degrees the index settles at the middle point of the scale, but we mark when the vibrations extend to equal distances on the opposite of its zero or middle point.

OF THE WHEEL AND AXLE.

247. The WHEEL AND AXLE consist of two cylinders of unequal radii, moveable about a common axis, with which they are invariably connected. The greater cylinder is called the WHEEL, and the less the AXLE.

The power is supposed to act at the circumference of the wheel, and the weight at that of the axle in a direction to produce the opposite rotation.

The power and weight are usually applied in the direction of tangents to circular sections of the two cylinders, and are, therefore, situated in planes which cut the axis at right angles (fig. 103).

The WINDLASS, OR WINCH AND BARREL, is a variety of the same mechanic power, composed of a small cylinder, around which the rope is wound, which sustains the weight or resistance, and a handle, to which the weight is applied.

The CAPSTAN is a windlass, furnished with a number of handles arranged like spokes around one end of the barrel. For greater convenience in the application of the power, the barrel is upright, and the handles are, therefore, in a horizontal plane.

248. The great value of the wheel and axle consists in its power of continuing the elevation, or other motion of the

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weight, through any desired distance. In using the lever for raising a weight, we are limited in height by the range of motion of the end of the weight arm; but with the wheel and axle, we may continue the operation as long as there is any rope to wind on the barrel.

249. The power and weight are applied in parallel planes, and not in the same plane. It is, however, more convenient to consider them as acting in the same plane, and we therefore proceed to show that this supposition does not alter their relations as to mechanical effect.

Let *B a* be a cylinder moveable round its axis, *C c* (fig. 103), and let any force *P* be applied tangentially at the point *A*. Draw the diameter, *C a*, and now suppose a force *Q* equal to *P* to be applied at *B* in an opposite but parallel direction. It is evident that the rotary power or moment of *Q* will be equal to that of *P*, and we may substitute *Q*, acting at *B*, for *P*, acting at *A*. Let *c a* be drawn parallel to *B A*, and suppose two forces, *R* and *S*, each equal to *P* or *Q*, to be applied at *a* tangentially and in opposite directions. They will obviously be parallel to *P* and *Q*. Lastly draw *B a*. Its intersection *M* with the axis *C c* bisects both these lines.

We have now three forces acting on the cylinder, viz., *Q*, *R* and *S*. But as the two latter are equal and opposite, they balance each other. Hence the three forces, *Q*, *R* and *S*, produce the same rotary effect as the single force *Q*, or, as before shown, as the single force *P*. We are, therefore, at liberty to assume the three forces, instead of the force *P*.

Compounding now the two parallel and equal forces, *Q* and *S*, which act the same way, we have their resultant, *T*, equal to their sum, and acting through the point *M*. Thus the three forces *Q*, *R*, *S* may be reduced to the two forces *T* and *R*. But the force *T* is applied directly at the axis, and, therefore, has no rotary power. It is balanced by the resistance at the pivots which hold the cylinder in its place. Therefore the three forces *Q*, *R* and *S* are reduced to the single force *R*, but they are equal in effect to the force *P*. Hence the force *R* is equal in its effect to the force *P*. That is, a force applied at any one point of the circumference of a cylinder may be shifted to any other point without a change of mechanical effect.

power arms are R and R_1, and whose weight arms are r and r_1.

Hence (by 236) $P \times R \times R_1 = W \times r \times r_1$.

That is, the power into the continued product of the radii of the wheels is equal to the weight into the continued product of the radii of the axles.

Second. By Friction. — In (fig. 108) the friction at m and m_1 is supposed to prevent the one surface from sliding on the other. Thus, whatever force may be excited by P at the end of r will be excited through m on the end of r_1, and so on. This machine, therefore, resolves itself into a combination of levers whose power and weight arms are respectively R R_1 and R_2 and r r_1 r_2. Therefore

$P \times R \times R_1 \times R_2 = W \times r \times r_1 \times r_2$ as before.

Third. By Teeth. — In (fig. 109) assuming the teeth to be so small in comparison with the radii of the wheels that their length may be neglected, we have, as before a combination of levers, the power and weight arms being R R_1 and r r_1. Therefore, $P \times R \times R_1 = W \times r \times r_1$.

As the circumferences of circles are proportional to their radii, we may express the equation of equilibrium thus.

The power into the continued product of the circumference of the wheels = weight into the continued product of the circumference of the axles.

When the wheels have equal teeth, their circumferences are proportional to the number of teeth. Hence the equation becomes the power into the continued product of the number of teeth of the wheels = weight into the continued product of the number of teeth of the axles.

256. OF THE FORM OF TEETH. — When the teeth are straight and long as in (fig. 110) the distance of the point of contact P from the centre B varies according to the attitude of the teeth in rotation. Thus as P moves upwards the point of contact recedes from B. In this way the force applied at A is made to act on B at a continually increasing leverage B P. Hence the power would require a constantly varying weight for equilibrium. To obviate the variable action of the interlocking teeth, they are formed with curved sides consisting of arcs of Epicycloids, (fig. 111).

The epicycloid is generated by the end of a string as it is unwound from a circle, (fig. 112). At each moment the unwinding string describes an infinitely small circular arc, with a radius R formed of the unwound part of the cord. Hence R at its termination, is always perpendicular to the curve, and it is also tangent to the generating circle.

Let now two wheels furnished with epicycloid teeth be supposed to touch at some point r. Draw through r a line at right angles to the touching surfaces. From what has just been said, it is evident that this line will be a tangent to both the wheels; that is, the point of contact r of the teeth will always be contained in a line tangent to the two wheels. As the mutual pressure of the wheels is exerted through r, and is at right angles to the surface of contact, it must act along $M\,N$. Therefore, the leverage with which the wheels act and react is $A\,M$ and $B\,N$, and these remain always the same. Hence the action is perfectly uniform.

OF THE PULLEY.

257. A straight cord acted on by any force P at one end will have equal tension excited throughout its length. Let it be bent around a smooth pin or roller A (fig. 113). Assuming its perfect flexibility, and supposing no friction where it touches the pin, the force P will still excite equal tension throughout its length, in all positions of the segments $M\,P$, $M\,W$. This contrivance enables us to apply a force P without loss in any convenient direction, instead of being limited to that which is directly opposite the weight or resistance. A rope thus bent around a pin or roller forms a single pulley.

In the single pulley, therefore, $P = W$.

258. **Of systems of Pulleys with a single continuous rope.**—The theory of this arrangement is illustrated by (fig. 114). There being 6 segments of the rope, each sustains a tension equal to $\frac{1}{6}\,W$. The power P is called on to balance this tension, and must therefore be equal to it. Hence $P = \frac{1}{6}\,W$ and $6\,P = W$. If n be the number of segments, then $n\,P = W$.

That is, the Power multiplied by the number of segments of rope is equal to the Weight.

Applying the above notation to the combinations represented in (figs. 115, 116) we have in (115) two segments of the rope, therefore $2\ P = W$. In (116) there are 6 segments of rope, therefore $6\ P = W$.

In the combination represented by (fig. 117), when the rope at P is drawn out and the weight raised, the different rollers revolve with different velocities, and therefore require to be mounted on separate pivots. More readily to compare these velocities, let the rollers be arranged as in (fig. 118). If now P be drawn out so as to raise W through the vertical height $M\,N$, each segment will be shortened by the quantity $M\ N$, and the whole length of rope passed over F will be equal to $6\ M\ N$. Over the roller A will pass a length $M\,N$, over B, $2\ M\,N$, over C, $3\ M\,N$, over D, $4\ M\ N$, over E, $5\ M\,N$, over F, $6\ M\ N$. Thus the rollers A, B, C, D, E, F will be compelled to revolve with velocities in the ratio of 1, 2, 3, 4, 5, 6.

259. In the system of pulleys represented in (fig. 119) the rollers are made of less and less diameters successively in the proportion of 6, 5, 4, 3, 2, 1, so that for a single rotation of each, they will deliver lengths of rope in this proportion. The time of rotation is thus made the same for all the rollers. Hence, they need not be separate, but may be formed of a single block on a single axis. This is White's pulley.

260. Of the Geometrical pulley. — In this there are as many distinct ropes as there are moveable pulleys. The geometrical pulley is of two kinds. The first is represented in (fig. 120.) In this the several ropes are attached by one end above at $L\ M\ N$, and at the other attached to the next moveable pulley, making thus two segments for each rope. B and A each have a tension $= \frac{1}{2}\ W$. Hence the whole force acting downwards on the pulley $C\ D$ is $\frac{1}{2}\ W$. Therefore tension of $D = \frac{1}{4}\ W$; and so tension of $F = \frac{1}{8}\ W$. In order to have equilibrium, therefore, we must have $P = \frac{1}{8}\ W$, or $8\ P = W$ or $2 \times 2 \times 2 \times P = W$. So for 4 moveable pulleys $2 \times 2 \times 2 \times 2 \times P = W$, &c. Hence, if n denote the number of moveable pulleys, the equation of equilibrium is $2^n\ P = W$.

That is, the weight is equal to the power multiplied by the number 2 raised to a power denoted by the number of moveable pulleys.

In the second form (fig. 121), each pulley has 3 segments of rope and we have $3 \times 3 \times 3 \times P = W$. With 4 moveable pulleys the equation becomes $3 \times 3 \times 3 \times 3 \times P = W$. Hence the equation is $3^n P = W$.

That is, the weight is equal to the power multiplied by the number 3 *when raised to a power denoted by the number of moveable pulleys.*

OF THE INCLINED PLANE.

261. This is any plane surface in a sloping position. Thus $A B$ (fig. 122) represents an inclined plane; the angle $A B C$ which it makes with the horizontal line $B C$ is its inclination, the vertical $A C$ is the height, and $B C$ the base of the plane.

In the equilibrium of a weight W maintained at rest on the inclined plane by the action of a power P, three forces are concerned, viz. the *weight*, the *reaction of the plane*, and the applied *power*. The directing of the first of these is always vertically downwards, and that of the second at right angles to the inclined plane, in an upward direction.

There are two principal cases of equilibrium on the inclined plane: first, when the power is parallel to the inclined plane; second, when it is parallel to the base.

262. *First case.* Let P act upwards along the plane parallel to $A B$ and in a direction passing through the centre of gravity of W the weight (fig. 122). As the reaction and weight also act through this point, we have three forces, viz., the power, weight and reaction in equilibrium at this point. Constructing the triangle $W H G$ with sides in the direction of these forces or parallel to them, we have (by triangle of forces) $P : W = G H : G W$. But by similar triangles $G H : G W = A C : B A$.

Therefore, $P : W = A C : B A$.

That is, *when the power is applied in a direction parallel to the length of the plane, the power is to the weight as the height to the length of the plane.*

263. *Second case.* — Let P act in a line parallel to the base $B C$ (fig. 123). Using the same vertical $W G$ to represent the weight, we have for the triangle of forces $W G I$.

Hence $P : W = G I : G W$. But by similar triangles $G I : G W = A C : B C$.

Therefore, $P : W = A\ C : B\ C$.

That is, *when the power is applied in a direction parallel to the base of the plane, the power is to the weight as the height to the base of the plane.*

264. It is obvious that for the same weight W, a greater power P is required when it is parallel to the base than when parallel to the length of the plane. The power in the two cases is in the proportion of $G\ I$ to $G\ H$; that is, in the proportion $A\ C$ to $B\ C$.

It will also be seen, that in the second case the reaction perpendicular to the plane is greater than that in the first in the ratio of $W\ I$ to $W\ H$.

265. If the two masses W and W_1 (fig. 124) placed on contiguous inclined planes $A\ B$, $A\ C$, be connected by a cord passing over a pulley at the vertex, the segments of the cord being parallel to the planes, there will be equilibrium when Power parallel to $A\ B$ = Power parallel to $A\ C$.

But Power parallel to $A\ B : W = A\ D : A\ B$; therefore $W = A\ B \times \frac{P}{A\ D}$. So also $W_1 = A\ C \times \frac{P}{A\ D}$. Hence

$$W : W_1 = A\ B : A\ C.$$

That is, *when two weights connected by a cord are in equilibrium on contiguous inclined planes, the weights are in the same proportion as the lengths of the planes.*

OF THE SCREW.

266. The screw is an inclined plane bent in a spiral form, or it may be regarded as a narrow inclined plane wrapped around a cylinder. The inclined surface or edge forms the thread of the screw, the base of the plane is folded around the base of the cylinder, and the height of the plane forms the height or length of the screw (fig. 125).

Taking a single turn of the screw, the length of the inclined plane forms the length of one turn of the thread, the base of the plane one circumference of the cylinder, and the height of the plane the interval between the two contiguous threads taken in the direction of the length of the cylinder or screw.

267. If a weight W be placed on the sloping spiral sur-

face (fig. 126) the conditions of equilibrium will be the same as if it were placed on a straight inclined plane of the same inclination; that is, on the inclined plane, which, by folding around the cylinder would form the spiral. If now we suppose the weight to be sustained on the spiral surface by a power applied parallel to the slope of the generating inclined plane, the conditions of equilibrium are those of the first case of the inclined plane, (262). Hence we have

$P : W =$ Height : Inclined length.

That is, in terms of the screw we have *Power is to Weight as the interval between threads is to the length of one turn of thread.*

Here the circumference of the cylinder is that described with a radius extending from the axis $A\ x$ to the point on which W rests. This proportion applies to the case of a weight or a vehicle drawn or sustained on a spiral road-way, by a rope parallel to the surface of the spiral.

268. When the power is applied in a direction parallel to the base of the spiral, that is, to the base of the generating plane, the conditions are those of the second case of the inclined plane, (263). We, therefore, have $P : W =$ Height : Base of plane.

That is, in terms of the screw, *Power is to Weight as the interval between threads is to the circumference of cylinder.*

269. If, instead of a weight, W, acting at a single point of the spiral, we have a number of weights at different points, or a weight or pressure of any kind distributed over the entire length of the screw-thread, but still every where parallel to $A\ x$ the axis of the screw, the above proportion will obviously hold for each element of the distributed pressure, and therefore for the whole. Such is actually the mode in which the pressure or resistance applies itself to the threads of a screw. (Fig. 127.)

270. In the common use of the screw, the direction of the power is parallel to the base of the cylinder, that is, of the generating inclined plane, as above assumed, we have therefore for the common screw, when the power is thus applied tangentially to the cylinder, the following proportion.

$P : W =$ interval of threads : circumference of cylinder.

Hence $P \times$ *circumference of cylinder* $= W \times$ *interval of threads.*

271. But for convenience as well as to increase the efficiency of the screw, the power is almost always applied at the end of a handle or lever, as at *B* (fig. 128), which is capable of turning in a circle at right angles to the axis of the screw, and therefore parallel to the base of the generating plane.

By removing the power from *W* to *P*, we increase its efficiency in the proportion of *M W* to *M P*, (by the principle of moments). Hence it will support a weight greater than that which it would support if applied at the circumference, in the same proportion. Instead, therefore, of having $P : W =$ interval of threads : circumference of cylinder, we must increase the last term in the ratio of *M W* to *M B*. But the circumference of the cylinder thus increased, becomes the circumference described by the point *B*, in one turn of the screw. Hence when the power is applied to a screw by a handle or lever, we have the following proportion, power : weight = interval of threads : circumference described by power.

This gives us for the equation of equilibrium of the screw,

Power × circumference described by power = Weight × interval of threads.

272. In each turn of the screw it advances or retreats according to the direction of the rotation by a distance equal to the interval of threads.

273. When the screw is formed as in the preceding instances, on the outside of a cylinder, it is called a male screw, when on the inside of a hollow cylinder, a female screw.

274. As in the screw, the efficiency of the power, that is, the weight which a given power is able to support, augments in proportion as the interval of the threads is less; it follows, that, in order to have great efficiency, a screw must have very slender threads, close together. But the liability of the threads to fracture sets a limit in practice to the increase of the efficiency of the screw in this way.

275. This disadvantage is obviated by *Hunter's screw*, which combines two screws of different intervals of thread, made to act in opposite directions, (fig. 129), the large screw *L*, passing through the fixed bar *A B*, has the smaller one *H* projecting from its extremity in the direction of its length.

This enters the block *a b*, which is moveable between the guides *A C*, *B D*. By one turn to the right, the large screw will move upwards a distance equal to one interval of the thread, say one inch, and will thus force the small screw upward the same distance. But this latter enters the block *a b*, which is furnished with a hollow screw to correspond, and is thus, in one turn, carried into *a b*, a distance equal to one interval of the thread of the smaller screw, say $\frac{3}{4}$ of an inch. The block *a b* will thus be moved upward, a distance $= 1$ inch — $\frac{3}{4}$ inch $= \frac{1}{4}$ inch. That is, the effect on *a b* is the same as if a single screw were used, having its interval of threads $\frac{1}{4}$ inch, in other words, the arrangements enables *P* to sustain a weight 4 times that which it could sustain by a screw one inch in diameter. If now *M* should differ from *L* by $\frac{1}{10}$ or $\frac{1}{100}$ of the interval of *L*, the efficiency of this combination would be 10 fold or 100 fold of that of the large screw.

Thus the relation of the power to the weight or resistance in the case of equilibrium, with Hunter's screw is as follows: power : weight or resistance = difference between the thread intervals of component screws : circumference described by the power. This gives for its equation of equilibrium the following:

Power × circumference of power = weight × difference of thread intervals of the two screws.

276. The same effect is produced when the smaller screw is made to penetrate the larger in a direction opposite to the advance of the latter, (fig. 130).

277. *The endless screw* is a short screw, made to work in the teeth of a wheel for the purpose either of giving or receiving motion. Each revolution of the screw obviously allows one tooth to pass, (fig. 131).

278. The *micrometer screw.*—This is a screw so arranged as to be used in making minute divisions on a line as in graduating the scales of various instruments. As in each complete turn, a screw moves in the direction of its length, through a distance equal to one interval of thread, so for $\frac{1}{2}$, $\frac{1}{10}$, or $\frac{1}{100}$ of a revolution, it moves through a corresponding part of an interval. The micrometer screw acts on this principle. The screw is one of very uniform and small thread, to the head of which is attached an index *x* (fig. 132),

revolving as the screw is turned in front of a graduated circle. The screw works in the frame *A B C D* pushing forward or drawing *a b* as it is turned. To *a b* is fastened a small point *s* at right angles to the plane of the figure, which, when pressed down, makes a fine mark on the metal plate on which the graduations are to be made. Suppose the interval of thread to be $\frac{1}{10}$th of an inch. Each complete turn of *x* will carry *s* forward a distance equal to $\frac{1}{10}$ inch, and this enables us to make on the plate beneath, a row of marks at intervals of $\frac{1}{10}$ of an inch apart. If now we wish them to be at intervals of any other given magnitude, say $\frac{1}{100}$th of an inch, we revolve *x* successively through an arc equal to $\frac{1}{10}$ of the circle, that is $\frac{360°}{10} = 36°$ and make a mark with the point at each of the stages.

OF THE WEDGE.

279. This mechanic power may be regarded as consisting of one or of two moveable inclined planes.

If a power *P* be applied to the back *A C* of an inclined plane *A B C* freely moveable in the direction *C B* on the horizontal surface *L M*, and a weight *W* press upon the surface *A B* through the medium of a vertical bar capable of being moved up and down between guides, the inclined plane becomes a wedge, (fig. 133). As the plane is moved forwards by the force *P*, *W* is forced upwards, and as it retreats *W* descends. The reaction of the plane is at right angles to the surface. Let this be denoted by *R*.

To find the relation of *P* to *W* in the state of equilibrium, observe that the effect of *P* to move *A B C* forward is the same at whatever point of *A C* this point may be applied. We may therefore regard it as being applied at *D* on the level with *O*, the point of application of *W*. Constructing the triangle *O I G* so that its sides shall be parallel to the weight, reaction and power, it becomes the triangle of forces.

Hence $P : W = I\,G : O\,G$ or by similar triangles $P : W = A\,C : B\,C$ that is, *Power* : *Weight* = *back of wedge* : *base of wedge.*

Again $P : R = I\,G : O\,I$, that is, *Power* : *Reaction* = *Back* : *Face.*

280. The wedge as commonly used resolves itself into two inclined planes united back to back, (fig. 139).

Let P be the power applied at C the middle of $A\,D$, and let R and R be the equal reactions or resistances acting perpendicularly at O and O_1 we may substitute for P at C $\frac{1}{2}P$ at M and $\frac{1}{2}P$ at N, then by (279) $\frac{1}{2}P : R = A\,C : A\,B$, and $\frac{1}{2}P : R = C\,D : B\,D$.

Whence $P : R + R = A\,D : A\,B + B\,D$.

That is, *the Power applied to back : Sum of resistances on the faces of the wedge = Back : Sum of Faces.*

281. TO PROVE THAT THE EQUATION OF EQUILIBRIUM OF ALL THE MECHANIC POWERS MAY BE REFERRED TO THE PRINCIPLE OF THE EQUILIBRIUM OF MOMENTS.

We have already deduced the equation of the lever and of the wheel and axle, from this principle. We may in like manner deduce that of the inclined plane, by considering the point of contact S of the body with the plane as the pivot or centre of moments, and drawing from it lines perpendicular to the directions of P and W.

1*st.* When the power is parallel to the plane (fig. 135), the perpendicular on P, that is, the lever arm of P, is the radius $S\,G$, for the radius is perpendicular to the tangent $A\,B_1$ and therefore to its parallel. Hence $P \times S\,G = W \times S\,M_1$ that is, by similar triangles $P \times A\,B = W \times A\,C$.

2*d.* When the power is parallel to the base (fig. 136), we have $P \times S\,N = W \times S\,M$, or by similar triangles $P \times B\,C = W \times A\,C$. It is obvious that this gives also the equation of the wedge and screw.

In the single pulley (fig. 137) the weight and power may be regarded as acting at the arms $C\,B$ and $A\,B$ of a lever of second order. Thus $P \times A\,B = W \times B\,C$ or $P \times 2\,B\,C = W \times B\,C$, that is, $2\,P = W$.

282. EXAMPLES OF THE COMBINATIONS OF THE MECHANIC POWERS. — *First.* In the machine (fig. 138) combining the endless screw, wheel and axle and pulley, we compute the relation of P to W as follows. $P \times$ circumference described by $P \times$ radius of wheel $\times$ number of ropes $= W \times$ interval of thread $\times$ radius of axle $\times 1$. Suppose the circumference described by power $= 30$ inches; interval of thread $= \frac{1}{10}$ inch; radius of wheel $= 2\frac{1}{2}$ inches; radius of axle $= \frac{1}{4}$ inch; number of segments of rope $= 4$. Then we have $P \times 30 \times 2\frac{1}{2} \times 4 = W \times \frac{1}{10} \times \frac{1}{4} \times 1$; that is, $P \times 300 = W \times \frac{1}{40}$, therefore $12000\,P = W$.

Second. In the (fig. 139) combining the wheel and axle, pulley and inclined plane, we have $P \times$ radius of wheel $\times$ number of ropes $\times$ length of plane $= W \times$ radius of axle $\times$ height of plane.

283. From what has been proved above, it follows that any combination of mechanism at the extremities of which P and W are in equilibrium may be represented by a single lever, whose arms are of such proportion as to cause equilibrium between the same P and W. Such a lever has been called an *Equivalent Lever.*

OF MOTIONS AND MOVING FORCES.

284. The preceding chapters, under the head of Rational Mechanics, include the principal elementary Theorems in Statics or the doctrine of the Equilibrium of Forces. These form the first part of Rational Mechanics. We now pass to the fundamental Theorems of Dynamics or the doctrine of Motions and Moving Forces, constituting the second part of that subdivision of Mechanical Philosophy.

CHAPTER XVI.

OF UNIFORM MOTIONS.

285. *Motions are either Uniform or Variable.* A uniform motion is one of unchanging velocity, or one in which equal spaces are described in equal successive times. A variable motion is one of changing velocity, or one in which unequal spaces are described in equal successive times.

286. In treating of Motions generally, the time is counted in seconds, the Velocity in feet per second, and the Space is the whole number of feet described in the given time.

287. In a uniform Motion, if we denote the Velocity of the Motion by V, the Time of its continuance by t, and the Space described in that time by S, it is evident that the relations of these three quantities are expressed by the equation $S = Vt.$

For as V feet is described in 1 second, t times as many feet, that is $V t$ feet must be described in t seconds.

From this expression for the space described we deduce the following for the Velocity and for the Time in Uniform Motions.

For the Velocity . . . $V = \frac{t}{S}$

For the Time . . . $t = \frac{S}{v}$

Thus in Uniform Motions the space is equal to the velocity multiplied by the time; the velocity is equal to the space divided by the time; and the time is equal to the space divided by the velocity.

288. We may represent graphically these relations of Space, Velocity and Time in uniform motions, by assuming a horizontal line, $A B$, (fig. 140,) to measure the time t, and a vertical $A C$ to measure the velocity V, using the same length in $A B$ to represent 1 second, as in $A C$ to represent 1 foot. Then completing the rectangle $A B C D$, its area will evidently be equal to $A B + A C$, and will, therefore, be equal to $V t$; that is, it will contain the same number of square feet as there are linear feet in S, the space described in the uniform motion. Dividing $A B$ at m, n, &c., and drawing the verticals $m p$, $n q$, &c., it is obvious that the spaces described in the times $A m$, $A n$, &c., with the same velocity V, will be measured by the number of square units in $A C p m$, $A C q n$, &c.; that is, by the areas of these figures.

Thus $A B$ and $A C$, and the rectangle which they form, represent the relations of the time, velocity, and space in uniform motions.

289. Of Forces which produce uniform motion.—It has already been shown that, in virtue of *inertia*, any motion is uniform when the moving body is unaffected by any mechanical force subsequent to that which impresses the motion. Thus a ball shot from a cannon would move uniformly, but for resisting and deflecting forces along its path. So a car on a horizontal railroad when once impelled, would move uniformly but for the friction, resistance of air, &c.

It is evident, however, that when the impelling force continues to act throughout the motion, it may be so proportioned to the resistances as just to balance their aggregate effect at each moment of the motion. Thus the railroad car, moving at the rate of 10 miles the hour, may be drawn by a force which is exactly equal to all the resistances. The tractional force will then be entirely consumed in balancing these resistances, and the car will move forward at the uniform rate of 10 miles per hour. In the same manner, a steamboat may be so impelled by the paddles that the whole motive power precisely balances the resistance of the water. In this case, the motion will be uniform.

290. We thus see that there are two conditions under which uniform motion may arise. *First.* When the mass once set in motion is completely isolated from all mechanical action. *Second.* When the mass being once in motion, the motive power is continued in just such intensity as is necessary to balance the resistances. In the latter condition there is equilibrium among the acting forces at the same time that there is motion. This is the state already described as that of Dynamic Equilibrium.

CHAPTER XVII.

OF UNIFORMLY ACCELERATED AND UNIFORMLY RETARDED MOTIONS.

291. VARIABLE motions are of two kinds. First, accelerated, and second, retarded motions. In the former the velocity continually increases, and in the latter it continually declines. A falling stone represents the one — a stone shot upwards, the other.

292. When the velocity of the moving body increases continually at an equal rate, that is, where for each equal addition to the time of the motion there is an equal addition to the velocity, it is a UNIFORMLY ACCELERATED motion. When for each equal addition of time there is an equal deduction from the velocity, it is a UNIFORMLY RETARDED motion.

293. The continual *acceleration* of a motion implies the continual action of a force upon the moving body in the direction of the motion. This accelerating force may be constant or variable in its intensity. If variable, the additions made to its velocity in successive seconds will be unequal. If the force be of constant intensity, these increments of velocity will be equal. In the latter case, the motion will be a uniformly accelerated motion.

Thus, *to produce a uniformly accelerated motion, the accelerating force must act incessantly and with unvarying intensity in the direction of the motion.*

294. The continual *retardation* of a motion implies the continual action of a force upon the moving body in a direction opposite to that of the motion. If this retarding force be of variable intensity, it will cause unequal decrements of velocity in successive seconds; but if the force be of constant intensity, it will cause equal decrements of velocity in successive seconds, and the motion will be uniformly retarded.

Thus, *to produce a uniformly retarded motion, the force must act incessantly and with unvarying intensity in a direction opposite to the existing motion.*

295. It will be proved hereafter that the weight of a body is a constant force, so long as the body remains near the earth's surface, and in the same latitude. Hence it causes a uniformly accelerated motion on the part of a falling body, and a uniformly retarded motion on the part of a body vertically projected upwards.

The laws of uniformly accelerated and retarded motions may, therefore, be deduced from experiments made with falling and ascending weights. This will form the subject of consideration under a later head.

296. If we call f the increment of velocity in $1''$ due to the constant accelerating force, this quantity will also represent the decrement in $1''$ due to the same force acting in the opposite direction.

Thus the weight of a falling body impresses upon it a velocity of 32 feet per second for every second in its descent; and, on the contrary, when the body is projected upwards, the same weight takes from its motion a velocity of 32 feet per second for every second of its ascent.

297. This quantity f is called *the acceleration*. It has already been shown that the acquired velocity is a proper measure of the intensity of the force. Hence, if there be two forces acting on equal masses, one of which, F, generates in 1″ the velocity f, and the other F generates in 1″ the velocity f_1, we shall have $F : F_1 = f : f_1$. In other words, constant accelerating forces are to one another in the proportion of the accelerations they produce in equal masses in 1″ or in any equal times.

298. Of the Velocity acquired in uniformly accelerated motion. — In a uniformly accelerated motion, the whole increment of velocity (v) in a given number of seconds (t), is equal to the acceleration in one second (f) multiplied by the number of seconds. For by the nature of the motion an equal addition is made to the velocity in each successive second. Hence, in 2″ the velocity will be $2f$, in 3″ it will be $3f$, in 10″ it will be $10f$, and so on. Hence the relation of velocity to time and acceleration is as follows:

$$v = f\,t.$$

That is, *the acquired velocity is equal to the acceleration multiplied by the time.*

299. Of the Space described in uniformly accelerated motion. — Supposing the accelerating force to act from the beginning, the velocity at the end of the first second is f feet per second, and the velocity at the middle of this second, that is, at the end of the first half second, is $\frac{1}{2}f$ feet per second. This is the average, or mean, of all the velocities, increasing from nothing to f during the first second. The moving body, therefore, would in that second describe the same space with this constant velocity as it actually does describe with the velocity accelerated uniformly from nothing to f. But in a uniform motion, space is equal to velocity multiplied by time. Therefore the space described in the first second is equal to half the acceleration due to that second; in other words, equal to half the space which the body would describe in the next second with the acquired velocity alone. Thus the velocity acquired in 1″ by a falling body is 32 feet per second; the space it falls through in the first second is, therefore, 16 feet.

300. Supposing the accelerating force to act from the beginning, the spaces described by the moving body in the 2d, 3d, or any other given second of the motion may be deduced from the above. Let $A\,x$ (fig. 141) represent the line of the motion, A the point at which it begins, and a the point which the body reaches at the end of the first second. Then the space $A\,a = \frac{1}{2} f$. The velocity of the body at a is f. Hence, if there were no acceleration during the next second, the body would describe the space $a\,x = f$, and at the end of that time be found at x. But the accelerating force acts through this second as through the first, and thus adds to the space $a\,x$ or f the further space $x\,b$ or $\frac{1}{2} f$. Hence, in the 2d second, the body moves through $3 \times \frac{1}{2} f$ feet. So again the body begins the 3d second with a velocity due to 2 seconds, of acceleration; that is, (298) with a velocity $= 2f$ or $4 \times \frac{1}{2} f$. This alone would in that second carry it to y. But the accelerating force acts during this as during the preceding second, and adds to the space described with the acquired velocity the further space $\frac{1}{2} f = y\,c$. Hence the whole space described in the 3d second, that is, $b\,c$, is equal to $4 \times \frac{1}{2} f + \frac{1}{2} f = 5 \times \frac{1}{2} f$. By extending the investigation to any succeeding seconds, we should have the same arithmetical series continued.

Hence the spaces described in the separate successive seconds, are in the proportion of the series of odd numbers.

In 1st second,	2d	3d	4th	5th	6th.
Space = 1,	3	5	7	9	11.

301. To compute the entire space described in 2, 3, or any number of continuous seconds reckoned from the beginning of the motion, we sum up the spaces described in the separate seconds as computed by (300).

Thus in 1″ the space described is . . $= \frac{1}{2} f$.
" 2″ $\frac{1}{2} f + 3 \times \frac{1}{2} f = 4 \times \frac{1}{2} f$.
" 3″ $\frac{1}{2} f + 3 \times \frac{1}{2} f + 5 \times \frac{1}{2} f = 9 \times \frac{1}{2} f$.
So in 10″ the space described will be $= 100 \times \frac{1}{2} f$.

Hence, in general, if S denote the space, t the time in which it is described, and f the acceleration or velocity due to one second, the relation of these quantities is given by the equation,

$$S = \tfrac{1}{2} f t^2.$$

Thus, in general, *to compute the space described in a given number of seconds, reckoned from the beginning of the motion, we multiply the space described in the first second by the square of the number of seconds.*

302. As long as the accelerating force remains of the same intensity, the acceleration f continues to have the same value. Hence, in the expression $v = f t$, (298,) we have f constant; and, therefore, v varies in proportion to t.

Therefore, in a uniformly accelerated motion, *the velocities acquired are proportional to the times in which they are acquired.*

303. So, likewise, in the expression $S = \tfrac{1}{2} f t^2$ (301), $\tfrac{1}{2} f$ is constant, and consequently S varies in proportion to t^2. Therefore, in a uniformly accelerated motion, *the spaces described from the beginning of the motion are proportional to the squares of times in which they are described.*

304. Examples.

(*a*). A ball dropped into a well is 5 seconds in reaching the bottom. Compute the velocity with which it strikes the bottom. Here we refer to the formulæ $v = f t$. We have $f = 32$ and $t = 5$ seconds. Therefore $v = 32 \times 5 = 160$ feet per second. This is the velocity with which the ball strikes the bottom.

(*b*). Again, in the same case, compute the depth of the well. This depth is S, the space described in 5 seconds by the falling body. Here we use the formulæ $S = \tfrac{1}{2} f t^2$. We have $\tfrac{1}{2} f = 16$ feet, and $t^2 = 5 \times 5 = 25$. Therefore the depth $S = 16 \times 25 = 400$ feet.

305. If when the constant *accelerating* force begins to act, the body be already moving with a velocity equal V, and the accelerating force continue to act for t seconds, the increment of velocity will be $v = f t$, and the whole velocity of the body at the end of t seconds will be $V t + f t$.

If the force be a constant *retarding* one, the decrement of velocity in the time t will be the decrement of 1″ multiplied into the number of seconds $= f\,t$. Hence, if as above, V be the uniform velocity of the body before the constant force began to act, we shall have for the velocity at the end of t seconds, $V - f\,t$.

306. To illustrate this formulæ by the vertical motion of heavy bodies, suppose, *first*, a heavy body shot downwards with a velocity of 200 feet per second, and that it is 5 seconds in the descent, what will be its velocity on reaching the earth? Here $V = 200$ and $f\,t = 32 \times 5 = 160$. Therefore whole velocity $= 200 + 160 = 360$ feet per second. *Second*, let the body be shot vertically upwards with the same velocity, 200 feet per second, what will be its velocity at the end of 5 seconds? Here velocity $= V - f\,t = 200 - 160 = 40$ feet per second.

307. To find the space described in t seconds, by a body whose velocity, by uniform acceleration, is increased from V up to $V + f\,t$ as in (305), we proceed as follows: We compute the space which the body would describe in virtue of this original velocity V alone. This in t seconds would be equal to $V\,t$. Next we compute the space which it would describe in t seconds in virtue of the uniform acceleration alone. This would be equal to $\frac{1}{2} f\,t^2$. Therefore as these motions coexist in the body, the whole space described in the time t is $V\,t + \frac{1}{2} f\,t^2$.

308. To find the space described in t seconds by a body whose velocity, by uniform retardations, is diminished from V down to $V - f\,t$; we have $V\,t$ = space described in t seconds, in virtue of the original velocity V acting alone; and $\frac{1}{2} f\,t^2$ = space through which the constant retarding force would move the body in t seconds in the opposite direction. Hence the space actually described in t seconds must be $V\,t - \frac{1}{2} f\,t^2$.

309. To illustrate these formulæ by examples, suppose as above, $V = 200$ feet per second, and $t = 5''$. Then, in the case of a body projected vertically downwards we have space $= V\,t \;\; \frac{1}{2} f\,t^2 = 200 \times 5 \times 16 \times 25 = 1000 + 400 = 1400$. When the body is projected vertically upwards with the velocity 200, to find the space through which it rises in 5 seconds we have space $= V\,t - \frac{1}{2} f\,t^2 = 1000 - 4000 = 600$ feet.

310. It is obvious from what has been clearly shown, that in a uniformly accelerated motion the acquired velocity is proportional to the time, and the space described to the square of the time. Therefore, the space described is proportional to the square of the velocity. From this relation we may deduce a formula for finding the acquired velocity in terms of the space described. Thus we have $f =$ velocity acquired in the first second, and $\frac{1}{2} f =$ space described in the first second. Let $s =$ space described in some unknown time, and $v =$ velocity acquired in describing s; then we have.

$$\tfrac{1}{2} f : s = f^2 : v^2.$$

$$\text{Hence } \tfrac{1}{2} f v^2 = f^2 s; \text{ and } v^2 = 2 f s.$$

$$\text{Therefore } v = \sqrt{2 f s}.$$

That is, to *find the acquired velocity in terms of the space, multiply the space by twice the acceleration in* 1″, *and take the square root of the product.*

Thus, as an example let $s = 400$ feet. The velocity acquired in falling through

s is $v = \sqrt{2 f s} = \sqrt{2 \times 32 \times 400} = 8 \times 20 = 160$ feet per second.

311. As the acceleration per second caused by gravity is in round numbers 32 feet, we have $2 f = 2 \times 32 = 64$. Hence $\sqrt{2 f s} = \sqrt{64 . s} = 8 . \sqrt{s}$. Thus to compute in feet the velocity acquired by a body in falling through a given space we multiply the square root of the space by the number 8.

312. Assembling all the formulæ which relate to uniformly accelerated and retarded motions, we have

First. When the accelerating or retarding force acts from the beginning of the motion

$$(A).\ v = f t. \quad (B).\ s = \tfrac{1}{2} f t^2. \quad (C).\ v = \sqrt{2 f s}.$$

Second. When the moving body has a velocity V prior to the action of the constant force.

(D). Velocity at end of t seconds $= V \pm f t$.

(E). Space described in t seconds $= V t \pm \frac{1}{2} f t^2$.

313. The above formulæ apply to all uniformly accelerated and retarded motions, f in each case standing for the special acceleration produced by the force in one second. When the accelerating or retarding force is that of weight or gravity, the letter g is used to denote the acceleration per second. This, as already stated, is about 32 feet per second. The above formulæ applied to gravity are therefore written

(1). $v = g\,t$. (2). $s = \frac{1}{2}\,g\,t^2$. (3). $v = \sqrt{2\,g\,s}$.

(4). Velocity at end of t seconds $= V \pm g\,t$.

(5). Space described in t seconds $= V\,t \pm \frac{1}{2}\,g\,t^2$.

314. Experimental Illustrations of the Laws of Uniformly Accelerated Motion.—Very satisfactory illustrations of the laws above demonstrated may be obtained by the use of Atwood's machine, or by the horizontal apparatus for acceleration.

(*a*). *Of Atwood's machine.* (fig. 142).

This arrangement consists of an upright frame 8 or 10 feet in height, surmounted by a grooved wheel, over which, by means of a light and very flexible thread, weights are suspended. The frame is graduated in front, and is furnished with moveable stages, through which the weights, in their descent and ascent, may freely pass. When equal weights are attached to the ends of the thread, they will be in equilibrium. But if a small additional weight be placed on one side, its gravitating force will give motion to its own mass and to the aggregate mass of the other weights. In this way a small force is made to accelerate a large mass; and as the force remains constant, the motion thus imparted is still a uniformly accelerated one. At the same time it is sufficiently slow to be capable of comparison and measurement at its different stages, on the graduated scale of the machine. In using it, reference is constantly made to a pendulum or metronome which marks the time by audible beats.

Excluding for the present the friction and other resistances, let W be the sum of the equal weights hung at the ends of the cord, and w the additional weight which produces the motion. The velocity in one second, that is, the acceleration f, is thus found. Were the motive force or weight w to act in giving motion to the mass w alone, the velocity in one second would be 32 feet; and the effect of the force would be measured by the momentum, $32 \times w$. But in Atwood's machine the same force is made to give motion to $W + w$. The velocity in this case is f and the momentum is $f\,(W + w)$. But the moving force being the same, these effects or momenta produced in the same time must be equal, (76), that is, $f\,(W + w) = 32\,w$. Therefore $f = \frac{32\,w}{W + w}$. That is, the acceleration in Atwood's machine is found by dividing 32 times the small weight by the sum of all the weights.

Example. Let $w = 1$ ounce and $W = 15$ ounces, then we have $f = \frac{32 \times 1}{15 + 1}$. That is, $f = \frac{32}{16} = 2$ feet per 1″. By substituting the value of f thus found in the formulæ $v = f\,t$ and $s = \frac{1}{2} f\,t^2$, we have in this particular case $v = 2\,t$ and $s = \frac{1}{2}\,2\,t^2 = t^2$.

(*b*). Of the Horizontal Accelerating Machine, (fig. 143). This consists of a smooth horizontal board with parallel grooves or strips adapted to the flanges of the wheels of a small car. To the board placed on a long table is fastened an upright, furnished with pulleys at B and C, around which is passed a cord, secured at one end to the car, and sustaining at the other a small weight w. The board $A\,B$ is graduated into equal divisions, 1, 2, 3, on the edge. The motion of the car along $A\,B$ is caused by the gravity of the small weight w, made to act horizontally through the cord. As this force is constant, the motion of the car must be uniformly accelerated. The value of f, the acceleration per second, is determined as in Atwood's machine. Let W be the weight of the car. $W + w$ will be the whole mass moved by the gravity of w. Hence, as before,

$$f.\,(W + w) = 32\;w. \quad \text{Therefore } f = \frac{32\,w}{W + w}.$$

Knowing the value of f in numbers, we may compute the velocity v or the space s described by the car in t seconds, from the formulæ 1 and 2.

315. In what precedes no account is taken of the force expended in overcoming the friction, the imperfect flexibility of the cord, and the resistance of the air, or that consumed in accelerating the pulleys.

It will hereafter be proved that the force of friction is not sensibly changed by a change of velocity, in other words, that it is a constant force. Hence its effect will be merely to diminish the acceleration f by a constant quantity, and the motion will still be uniformly accelerated. In regard to the motion of the pulleys, it is obvious that they act in diminishing the acceleration due to w in the same way as an additional weight on the car. But as this is always the same, it will reduce the acceleration f by a constant quantity, and hence the motion will be uniformly accelerated.

But while neither the friction nor force consumed in moving the pulleys affects the uniformity of the acceleration, it is otherwise with the resistance of the air and the imperfect flexibility of the cord. Both of these resistances vary with the velocity, and therefore diminish the acceleration f by a variable amount. Fortunately for the practical use of the machines above described, the resistances last mentioned are of little energy compared with the friction and the force consumed in moving the pulleys; and therefore these arrangements may be regarded as furnishing a close approach to uniformly accelerated motion.

CHAPTER XVIII.

OF MOTION ON AN INCLINED PLANE.

316. It has been proved that when the power P is applied parallel to the length of an inclined plane, in order to prevent the descent of a mass, or weight W along the plane, we must have the proportion $P : W =$ height $=$ length of plane. In other words, P must be the same fraction of W, that the height is of the length. Hence so long as the inclination of the plane remains the same, P will have a constant ratio to W.

But the force P which balances the force with which the body tends to descend the plane must be equal to that force.

That is, the accelerating force down the plane is equal to P. And this force is constant while the inclination remains the same. Therefore *the motion down an Inclined Plane is a uniformly accelerated motion.*

317. If a body be set in motion up the plane, the constant force P will retard the motion by equal decrements of velocity in successive seconds. Hence *the motion of a body ascending an Inclined Plane is a uniformly retarded motion.*

318. *To find the value of the acceleration f on the Inclined Plane*, we proceed as follows. Since f is the velocity produced in 1″ by the accelerating force P acting on the mass (fig. 144) and g or 32 is the velocity produced in 1″ by the gravity W on the same mass; we have,

$$f : 32 = P : W.$$

But $P : W =$ Height : Length.

Therefore $f : 32 =$ Height : Length.

$$\text{Hence } f = 32 \times \frac{\text{Height}}{\text{Length}}.$$

That is, the acceleration per second on a given Inclined Plane is found by taking 32 times the height divided by the length of the plane.

Let h denote the height and l the length of an inclined plane, and let f be the acceleration proper to this plane, then we have $f = g \times \frac{h}{l}$.

319. To compute the velocity v acquired in t seconds and the space s described in t seconds by a body descending the plane, we substitute the above value of f in the equations A B and C of (312). We then have

$$(1).\ v = g\frac{h}{l}t. \quad (2).\ s = \tfrac{1}{2}g\frac{h}{l}t^2. \quad (3).\ v = \sqrt{2g\frac{h}{l}s}.$$

320. Examples. 1st. An inclined plane is 200 feet in height and 800 feet in length; required the velocity of a descending body at the end of 5″ and the space described in 5″. Here $f = 32 \times \frac{h}{l} = 32 \times \frac{200}{800} = 32 \times \frac{1}{4} = 8$, that is, the acceleration per second $= 8$ feet. In the formula $v = f\,t$, we put 8 for f and 5 for t. Hence $v = 8 \times 5 = 40$ feet

per second the acquired velocity. To compute the spac
have $s = \frac{1}{2} f t^2 = \frac{1}{2} 8 \times 5^2 = 4 \times 25 = 100$ feet.

2d. A railroad has an inclination of one degree; tha
92 feet in one mile, or in 5280 feet; required the accelera
per second of a descending car, also the velocity and s
corresponding to 1 minute or 60″. Here we have
$32 \times \frac{92}{5280} = 32 \times \frac{1}{57} . = \frac{32}{57}$ feet per second ne
The acquired velocity $v = f t = \frac{32}{57} \times 60 = 33.7$ feet
second. The space described $= s = \frac{1}{2} f t^2 = \frac{1}{2} \times \frac{32}{57} \times$
$= \frac{16}{57} \times 3600 = 1018$ feet.

3d. An inclined plane is 200 feet high and 800 feet l
and a body is set in motion up the plane with a velocit
80 feet per second; to find its velocity at the end of 6″
the space it has described in that time. Here $f = 32 \times$
$= 80$. As in this case the motion is uniformly retarde
have, Velocity $= V - f t = 80 - 8 \times 6 = 80 - 48 =$
Space $= V t - \frac{1}{2} f t^2 = 80 \times 6 - \frac{1}{2} 8 \times 6^2 = 480 -$
$= 336$ feet.

321. In order to compare the velocity acquired by desc
ing from A to B on the inclined plane with that acq
by descending from A to C in the vertical, we refer t
formula $v = \sqrt{2 f s}$. In regard to the vertical motio
have the acceleration $f = g$, and in regard to the m
down the plane $f = g \times \frac{h}{l}$. Hence velocity at

$B = \sqrt{2 g \frac{h}{l} \times l} = \sqrt{2 g h}$, the l's cancelling; an
locity at $C = \sqrt{2 g h}$, the same as the preceding.

Thus the velocity acquired by descending the plane is
to that acquired by falling through its vertical height.

322. If there be any number of inclined planes, $A B$,
$A E$, having the same or a common vertical height $A C$
have by the preceding proposition, the velocity at B, a
at E, &c., each be equal to the velocity at C, and henc
velocities at B, D, E, &c., are equal to one another. T
therefore, *whatever be the length or inclination of the*
by which a body descends through a given vertical heig
velocity on reaching the lower horizontal line will alwa
the same; and will be equal to that acquired by fa
through the vertical height. (Fig. 145.)

323. Let a body descend from A along a series of connected inclined planes, $A\,B$, $B\,C$, $C\,D$, (fig. 146,) and let the horizontal lines $B\,m$, $C\,n$ and $D\,E$ be drawn to meet the vertical $A\,E$. The velocity at $B =$ velocity at m by (331.) Supposing no loss of velocity by impact at B, the body would begin its motion on $B\,C$, and its descent in $m\,n$ with equal velocity. But (331) the velocity acquired by descending from B to $C =$ velocity acquired by falling from m to n. Adding these to the equal velocities acquired at B and m, we have the whole velocity at $C =$ whole velocity at n. In like manner the velocity at $D =$ velocity at E.

Thus the velocity acquired by descending a series of inclined planes, is equal to that acquired by falling through their perpendicular height.

324. Since the curved line $A\,B$ (fig. 147) may be regarded as made up of an infinite number of infinitely short right lines; the curved surface which it represents may be viewed as a series of infinitely short inclined planes; and, therefore, by (323) the velocity at D on the curve $=$ velocity at m on the same level in the vertical; and so velocity at $B =$ velocity at C. *Thus the velocity acquired by descending a curve is equal to that which is acquired by falling through its vertical height.*

325. Uniting what has been proved above, (322, 323, 324,) we see that the velocity which a body acquires by descending from a given point to a lower level, is independent of the path pursued by the body, being the same for a single inclined plane, a series of inclined planes, or a curve; and that this velocity is the same as that which the body would acquire by falling vertically from the given point to the given lower level. Thus (fig. 148) the velocity will be the same whether the body has descended $A\,B$, $A\,C$, $A\,D$ or $A\,E$.

Hence, to compute the acquired velocity, it is only necessary to know the vertical fall corresponding to the motion, and put it for s in the formula $v = \sqrt{2\,g\,s} = 8\sqrt{s}$.

Example: A car moved by gravity along an undulating but descending railway has reached a point 144 feet lower than that from which it started, required its velocity.

Here $s = 144$ and $v = 8\sqrt{144} = 8 \times 12 = 96$ feet per second. (Fig. 149.)

326. *Of the time of descending an inclined plane.* — T may be compared with that of falling through the verti thus: Referring to the formula $v = f\,t$, we have velocit B (fig. 150) $= f \times$ time down the plane, and velocity $C = g \times$ time down the vertical. But by (321) veloc at $B =$ velocity at C. Therefore $f \times$ time down the pl $= g \times$ time down the vertical. Hence time down plane : time down the vertical $= g : f = A\,B : A\,C$.

That is, the time down the plane is to the time down vertical as the length of the plane is to its vertical height.

327. In order to compare the spaces described in eq times by bodies descending the inclined plane and the ve cal respectively, we draw from any point M in the verti the line $M\,N$ perpendicular to the plane. Since $A\,M$, A $M\,N$ are severally parallel to the force of gravity, the fo down the plane and the reaction of the plane, $A\,N\,M$ is triangle of forces, and we have the proportion $A\,M : A\,N$ weight : force down the plane. But in order that the spa described may be proportional to the forces which ca them to be described, the times must be equal. Hence time down $A\,N =$ time down $A\,M$. So if $C\,K$ be dra perpendicular to $A\,B$, we have the time down $A\,K =$ ti down $A\,C$. *Thus the spaces described in equal times on plane and in the vertical, are defined by a perpendicu drawn from the vertical to the plane.* (Fig. 151.)

328. It follows from the above, that if in a vertical cir $A\,K\,C$ (fig. 152) we have a series of inclined planes A $A\,K_1$ forming chords diverging from A the summit of t vertical diameter, and another series, $C\,K$, $C\,K_1$, convergi to C the lowest point, *the times of describing all these chor are equal.* For the angles at K and K_1 by a property the circle, are right angles; therefore, by (337) we have ti down $A\,K =$ time down $A\,C$, and time down $A\,K_1 =$ ti down $A\,C$. Therefore time down $A\,K =$ time down A and so of all chords diverging from A. Next, complete t rectangles $K\,V$ and $K_1\,V_1$. Then, since $C\,K$ is equal a parallel to $A\,V_1$, time down $K\,C =$ time down $A\,C$, and likewise, time down $K_1\,C =$ time down $A\,V_1 =$ time do $A\,C$. Therefore time down $K\,C =$ time down $K_1\,C$, and for all other chords meeting in C.

CHAPTER XIX.

OF WEIGHT OR GRAVITY.

329. We have already had occasion to speak of the force of weight or gravity, in treating of equilibrium and of moving pressures. We have also considered in detail the accelerated and retarded motions, of which it is the cause, as well by its single action in the vertical direction, as by its combination with another force in the motion of bodies on an inclined plane. We are now to consider the variations of gravity as dependent on the mass of the gravitating body, its distance from the earth, and its position on the earth's surface.

LAWS OF GRAVITY.

330. Law I. The force of Gravity is proportional to the quantity of matter in the gravitating body. — The evidence of this law is derived both from Statical and Dynamical phenomena.

First. — *Statical proof.* Let M be a given bulk, say 1 cubic foot of any homogeneous material, for instance, iron, and let N be a volume of the same material twice as great. It is obvious that the quantity of matter in $N =$ twice the quantity of matter in $M = 2\,M$. It is also clear that the gravity of $N =$ twice the gravity of M, both being measured by the compression of a spring or other statical measure. Thus let M be suspended on the lever (fig. 153) at A_2 so as to compress the spiral spring $m\,n$; after marking the amount of the compression, remove M and place N on the level at A_1, half the distance from the fulcrum C we shall find the compression the same as before. But to produce the same effect with half the leverage, the force acting must be doubled. Therefore the gravitating force of $N =$ twice the gravitating force of M. Thus the gravity of N is to the gravity of M in the same proportion as the quantity of matter in N to that in M. The same may be proved for any other ratio of N to M. Thus it is clear that in regard to bodies made of the same material, the gravitating forces are proportional to the q tities of matter.

331. It must be borne in mind that this statical pro the law is in strictness not applicable where M and N sist of *unlike* matter. Where, as above, they are supp to be of the same kind, there can be no error in assu that the double volume contains a double quantity of ma But if we suppose another body, O, consisting of a diff material, say *glass*, equal in weight to the mass N, or t M of *iron*, we have no right to infer that it contains t the quantity of matter contained in M. We might ima the glass, O, to consist of particles, each having less g tating force than the particles of iron, and yet each equ matter to a particle of iron. In this case, to have an e force of gravity, the quantity of matter in the glass w be greater than that in the iron. Nothing heretofore pr contradicts this supposition. To demonstrate the law regard to heterogeneous bodies, we must resort to the tions produced by gravity.

332. Second. *Dynamic proof.* — It has already shown (76), that to compare one moving force with anot we compare the momenta which they respectively caus acting with uniform intensity for equal times.

Thus let Q be a certain quantity of matter and Q_1 s greater quantity, and let a force F acting uniformly o for a given time impress upon it the velocity V, and l force F_1 acting uniformly on Q_1 for the same time imp upon it the velocity V_1. Then $Q\ V$ and $Q\ V_1$ are the menta caused or the work done by the two forces in given time, and therefore $F : F_1 = Q\ V : Q_1\ V_1$. Now if velocities V and V_1 impressed in the given time by the forces be equal, we will have $F : F_1 = Q : Q_1$. That the moving forces will be proportional to the quantitie matter moved.

333. Observation and experiment prove that all bodie the earth's surface, however different in volume and terial, receive an equal acceleration if permitted to desc for the same time in an unresisting space. Thus a lea bullet, a pith ball, and a fragment of down dropped at same moment from the top of a vessel emptied of air,

strike the bottom at the same time and with the same velocity. Applying the principle above stated, we infer, that the gravitating forces impelling the bullet, pith ball, and down, are proportional to the quantities of matter in these bodies respectively. Reasoning similarly in regard to all other cases, we infer the truth of the general law, that the force of gravity is proportional to the quantity of matter in the gravitating body, and this, whether the bodies compared be like or unlike.

334. The fact of equal acceleration may be illustrated without using a vacuum, by employing dense bodies, such as lead, glass, &c., when it will be seen that their acceleration, even in falling through the air, is very nearly equal. When, however, we compare the acceleration of one of these with that of a light body, as a mass of lead with a feather, we find the acceleration of the latter much the less. This arises from the light body *exposing a larger surface in proportion to its mass* to the resisting action of the air. If, however, we place a light disc, as of paper upon a disc of brass or lead, we find that the paper falls as fast as the metal. Here the resistance of the air, acting below the metallic disc, is distributed in its action between this and the paper, just in proportion to the matter in each. Hence it produces no greater diminution in the acceleration of the one than of the other.

335. In what has been said above, the force of gravity has been regarded as a single force drawing the body towards the earth in the vertical direction, that is, in a direction at right angles to the earth's surface. Considering the earth as a sphere, every vertical line thus defined must pass through its centre. Therefore, the force of gravity, regarded as a single force acting on each mass at or near the earth's surface, has a direction passing through the centre of the earth.

336. But the gravity acting upon any mass as P (fig. 154) is not in reality a single force. It is the resultant of the separate attractions exerted upon P by every particle of matter in the earth. This attraction of the parts of the earth separately for a body near its surface, is shown by numerous facts, such as the following: —

(*a*). A plummet suspended near the side of a high mountain (fig. 155), at *A* or A_1 will not hang in the direction *A C* or A_1 *C* towards the centre, but in the line *A M* or A_1 *M* intersecting the middle radius at a point *M* above the centre. This has been proved by observations on Chimborazo, Mount Schehallion in Scotland, &c.

(*b*). A small mass, *P*, (fig. 156), suspended by a method hereafter to be described, and brought near to a large globe of lead or other dense substance, is observed to be attracted by it.

We, therefore, conclude that every particle of matter in the earth attracts the body *P* (fig. 154), and as each particle in the body *P* is drawn downwards, we may affirm that every particle in the earth attracts every particle in the mass *P*. By the law of action and reaction, this force is mutual, that is, the particles of the gravitating body attract the particles of the earth as much as the particles of the earth attract those of the gravitating body.

337. To show why the resultant of these molecular attractive forces must have a direction *P E* (fig. 154) passing through the centre of the earth, we assume any layer of particles *A B* at right angles to *P E*, and take two particles, *x* and *y*, similarly placed with respect to *P E*. Draw *P x* and *P y*. As *P* is equidistant from *x* and *y*, the force *P m* is equal to the force *P n*, therefore the resultant *P r* lies midway between *P m* and *P n*, and consequently is directed towards *E*. The same may be proved of every other pair of particles similarly placed on the opposite sides of *P E*. Hence the resultant of all the attractions of the layer *A B* is directed to the centre *E*. But the whole sphere may be considered as made up of layers parallel to *A B*, and the resultant force of each of these layers will be in a line passing through *E*. Hence the resultant force of all the particles of the sphere, that is, the whole gravitating action of the earth upon the mass *P*, must be in a direction passing through the centre.

338. The above reasoning supposes the earth to be not only an exact sphere, but of density which is either equal throughout or equal every where at the same distance from the centre. In the latter case, (fig. 157), the layer *A B* would be made up of parts having equal density at equal

distances on the opposite sides of $P E$, and therefore would still have a vertical resultant.

The irregularities of the earth's surface, and its division into seas and continents, the material of the latter much denser than that of the former, will, of course, slightly modify the direction of the resultant gravity. This effect will vary in amount according to the locality. At the seashore the plummet will tend slightly towards the land, and near a great mountain chain it will be deflected towards the mountain, as already shown. But these irregularities of form and density extend to so small a depth compared with the earth's radius (nearly 4000 miles), that their disturbing influence can only be detected by the most refined arrangements.

339. Regarding the earth as a sphere constituted as above described, and neglecting the trifling irregularities of surface which have just been noticed, it is obvious that the resultant gravity acting on P must be the same in quantity, and have a central direction for every point of the earth's surface.

340. But there are two causes not yet considered, which have a much more important influence upon both the direction and the amount of the force of gravity at different points of the earth's surface. These are —

1st. *The earth's form being that of a spheroid flattened at the poles.*

2d. *The rotation of the earth on its axis.*

The effect of these causes will be more fully discussed hereafter. For the present it is enough to see, that when the body is placed at one of the poles of the earth, as P (fig. 158), or at the equator Q, its gravity will be directed to the centre E. But when it is placed at any other point, as P_1, its gravity still vertical or perpendicular to the surface, will not be in a direction passing through E.

341. Law II. THE FORCE OF GRAVITY DIMINISHES AS WE RECEDE FROM THE EARTH, IN THE SAME PROPORTION AS THE SQUARE OF THE DISTANCE FROM THE EARTH'S CENTRE INCREASES.

The diminution of the weight of a body as it is carried higher above the earth's surface, is shown by the slower vibration of a pendulum on the summit of a high mountain than at its base. It might also be proved by hanging from a very delicate spring-balance a given mass, and then carrying it to a great height. It would be found to produce less compression of the spring as it was carried higher. The actual diminution of gravity, however, is very small at any height accessible to man. At the height of one mile, gravity is diminished by about $\frac{1}{2000}$ of the whole force at the earth's surface. That is, a body weighing 2000 lbs. at the surface, would weigh 1999 lbs at the height of one mile.

342. In order to compare the gravitating force at great distances with that at the earth's surface, recourse must be had to astronomical phenomena.

The motion of the moon around the earth affords us such a means of comparison. Let AMB represent the moon's orbit around the earth. (fig. 159.) The radius of this orbit is known to be nearly 60 radii of the earth. That is, $EM = 60\,ES$. Now the attraction exerted by the earth upon the moon is what causes the moon to move in the curve MB instead of a straight line Mt. Let Mx be the distance described in the orbit by the moon in 1 minute, xy is evidently the distance through which gravity has drawn the moon in that time from the line Mt in which it would otherwise have moved. Knowing the orbit and velocity of the moon, we can compute the length of this line xy, fallen through by the moon, in one minute. The result is $xy =$ 16 ft. Now, at the earth's surface, the force of gravity would cause a body to fall through $16 \times 60^2 = 3600 \times 16$ ft. Thus, the gravity at the distance of the moon's orbit from the earth is only $\frac{1}{3600}$th of gravity at the earth's surface. But the distance of the moon's orbit from the centre of the earth is 60 times the distance of the surface from the centre.

We thus see, that by increasing the distance 60 fold the force of gravity is reduced to $\frac{1}{3600}$th.

343. By similar observations in regard to the motions of the earth and planets around the sun, and of the satellites around their primaries, it may be demonstrated, that the laws of gravity above stated, obtain throughout the solar system, namely, that each planet gravitates to the sun with a force proportional to the mass of the planet directly, and the square of the distance from the sun inversely; and the same of the gravity of the satellites towards their primaries.

The proof of the laws of gravity as thus extended must be deferred until we treat of the motions of the planets.

344. An attempt has been made to prove the second law of gravity independently of observation, by assuming that gravity acts in right lines diverging in every direction from the centre of force. Thus let C (fig. 160) be the centre of this radiating force. Imagine a rectangular surface $R\ S$ at a certain distance, $R\ C$, from the centre, and another similar surface $P\ Q$, at half the distance. It is obvious that all the lines which can be drawn from C to $R\ S$ must pass through $P\ Q$. There must, therefore, be the same number of gravitating forces acting on $P\ Q$ as on $R\ S$, and assuming the force in each line to remain constant, the whole gravity exerted on $R\ S$ must equal the whole gravity exerted on $P\ Q$. But since $R\ C = 2\ P\ C$ we have (by geometry) $R\ S = 4\ P\ Q$. Let $a = \frac{1}{4}\ R\ S = P\ Q$. Then Gravity acting on a at the distance $R\ C = \frac{1}{4}$ Gravity acting on $R\ S = \frac{1}{4}$ Gravity acting on a at the distance $P\ C$.

Thus, the same surface a taken to a double distance, will have the gravity reduced to one-fourth. So at a triple distance it would be reduced to one-ninth, and in general the force is proportional inversely to the square of the distance.

345. In the above proof it is assumed that every line which can be drawn from C to the surface $R\ S$ is a line of gravitating force. But this cannot be true, unless every line thus drawn terminates in a particle of matter in $R\ S$; for if the line $C\ n$, for instance, end in a space between particles, there can be no gravity from n towards C, because there is nothing at n to be attracted. But it has been proved under a former head, (139,) that a large part of the bulk of even the densest bodies consists of interstitial space. Therefore we are not at liberty to assume that every line drawn from the gravitating centre C to the given surface $R\ S$ is a line of gravitating force. Hence this mode of proof must be abandoned as unsound.

346. It may be proved that, assuming all the particles x x_2 &c. of the sphere $A\ B$ (fig. 161) to act upon an external particle P according to the square of the distance inversely, the action of the whole sphere will be the same as if all its matter were collected at the centre; that is, the force will be proportional directly to the mass and inversely to $P\ C^2$.

It is also capable of being proved that when the attrac mass is a spheroid (fig. 162) instead of a sphere, and eac its particles attracts the external particle according to square of the distance inversely, the action of the w spheroid will be the same as if the whole mass were colle at the centre.

From this we deduce a simple rule for comparing gravity of P towards the spheres A and B (fig. 163), posing each sphere of uniform density, throughout. combining the two laws of gravity we have the follo proportion.

$$\text{Gravity to } A\ B : \text{Gravity to } a\ b = \frac{Mass\ A\ B}{R^2} : \frac{Mass}{r^2}$$

347. If now the spheres be of the same density, masses will be as R^3 to r^3. Hence the Gravity to A Gravity to $a\ b = \frac{R^3}{R^2} : \frac{r^3}{r^2} = R : r$. That is, *the gra at the surface of spheres of equal density are proport to their radii.*

The radius of the moon is 1076 miles, and the mean ra of the earth is 3963 miles. The mean density of the m of the moon is $\frac{56}{100}$ths that of the earth. Supposing the spheres to have the same density, the force of gravity a moon's surface would be to that at the earth's surfac 1076 to 3963, or as 1 to 3.68. But the force at the mo surface is only $\frac{56}{100}$ths of the value here given. This for therefore to the gravity at the earth's surface as $\frac{56}{100}$ to that is, as 1 to 6.57. Hence a falling body at the mo surface would acquire per second the velocity $\frac{g}{6.57}$. is, at the moon's surface $f = \frac{g}{6.57}$.

The radius of the sun is 441,000 miles; that of the e 3963 miles, and the density of the matter of the sun is fourth of that of the earth. Supposing them to have same density, the force of gravity at the sun would be to at the earth as 441.000 to 3963, or as 111 to 1. Hence, ta account of the less density of the sun, which reduces force of gravity at his surface to one-fourth of this, we force at the earth's surface, to force at surface of sun as to 1. Hence, at the surface of the sun $f = 27.7\ g$.

348. Of gravity within a hollow sphere. Let $A\ B\ a\ b$ (fig. 164) represent a spherical shell consisting of a single layer of particles throughout, and let P be a particle or small body any where within the cavity. It is obvious that P will be attracted in all directions, and that if it be at the centre of the shell, these attractions will all be equal, and P will remain poised at the centre. To compare the effects of the opposite attractions acting on P when out of the centre, conceive the lines $A\,b$, $B\,a$, $C\,c$ to be drawn passing through P, forming the edges of two triangular pyramids, whose bases $A\ B\ C$ and $a\ b\ c$ are indefinitely small parts of the spherical shell. Applying the two laws before demonstrated, we have the following proportion: —

$$\text{Attraction of } A\ B\ C : \text{Attraction of } a\ b\ c = \frac{A\ B\ C}{P\ B^2} : \frac{a\ b\ c}{p\ b^2}.$$

But (by Geometry) $A\ B\ C : a\ b\ c = P\ B_2 : P\ b^2$. Hence,

$$\text{Attraction of } A\ B\ C : \text{Attraction of } a\ b\ c = \frac{P\ B^2}{P\ B^2} : \frac{p\ b^2}{p\ b^2}.$$

$$\text{Therefore, Attraction of } A\ B\ C = \text{Attraction of } a\ b\ c.$$

That is, the attraction exerted on P by the corresponding parts of the shell on opposite sides of P are equal, the more remote side compensating for the diminished force due to its greater distance by its larger quantity of matter.

As the whole shell may be resolved into such pairs of antagonizing triangles, it follows that whatever may be the position of P within the thin shell, it is attracted equally in all directions. Since the shell of uniform thickness and density (fig. 165) may be regarded as made up of concentric shells, each like the above, having the thickness of one particle, and as the action of each of these shells upon P is equal in all directions around, it follows that their action when united to form the shell $A\ B\ a\ b$ is equal in all directions. Therefore, *a body P placed any where in the interior of a spherical shell of uniform thickness and density will be equally attracted in all directions.*

349. From this it may be inferred, that if the earth were a hollow sphere as has been imagined by some, the motions of bodies in the interior would be independent of gravity. A

stone released from the hand would not fall to the ground, and when on the ground it would not press upon it. Water would not flow; a body thrown upward with even a small force would continue to move uniformly across the cavity until arrested by the other side.

350. OF GRAVITY BELOW THE EARTH'S SURFACE. If we suppose a body to be placed within the earth at x (fig. 166), it is obvious that the attraction of that portion of the earth which lies between it and the surface acts in opposition to the general gravity, rendering the effective weight of the body towards the centre less in proportion as it is nearer to that point.

To compare accurately the weight of a particle at x with the weight at A, we conceive a spherical surface immediately external to x, and having $C x$ for its radius. This divides the globe into an external shell $M\ m\ N\ n$ and an inner sphere. The action of the shell upon the particle or body at x is neutral by (348), and hence we may regard the body as acted on exclusively by the attraction of the inner sphere $m\ C\ n$. We have therefore to compare the force of the whole earth acting on the body at A, with the force of the inner sphere acting upon it at x.

Assuming the earth to be of uniform density throughout, and applying the principle of (347), we have weight at A : weight at $x = A\ C : x\ C$.

That is, the weight of a body placed at the earth's surface is to its weight, when placed any where below the surface, as the radius of the earth is to the distance of the interior point from the centre.

351. We thus see that the weight of a body diminishes as it is removed from the earth's surface, downwards as well as upwards, although according to a very different rate of variation. Evidently a body has its maximum weight at the surface. At the distance of 3000 miles below the surface, or 1000 miles from the centre, its weight would be reduced to $\frac{1}{4}$. At the distance of 4000 miles above the surface it would be reduced to $\frac{1}{4}$. At the depth of 4000 miles, that is, at the centre, its weight would be 0.

352. DETERMINATION OF THE QUANTITY OF MATTER IN THE EARTH. — On the principles above explained are founded the

methods of determining the quantity of matter in the earth. These are of two kinds, —

First. By comparing the attraction of the earth with that of a sphere of lead or other dense material of known mass and dimensions.

Second. By comparing the attraction of the earth with that of a mountain of known mass and dimensions.

353. *First Method.* This may be explained by reference to (fig. 167), which represents the essential parts of the apparatus used by Cavendish. The large balls of lead *A* and *B* are placed on a strong platform moveable around a vertical pivot. Above is a slender horizontal bar *R T*, from the ends of which are suspended by rigid wires the small balls *P* and *Q*. The horizontal bar is supported by two oblique wires *R S* and *T S* attached at *S* to a slender vertical wire, which is firmly clamped at its point of suspension, so that a rotation of *R T* will twist the vertical wire, but will not cause it to turn in its fastening at the top.

The elasticity of the vertical wire causes the rod *R T* to vibrate when disturbed from its position of rest, and the amount of deviation is the measure of the disturbing force. The rod being at rest, the large balls are brought into the position represented in (fig. 168.) This causes the action of *A* upon *P*, and that of *B* upon *Q* to be in the same direction of rotation. The rod is at once deflected from its position of rest, and the wire twisted until the elastic reaction of the latter becomes sufficient to balance the attraction of *A* and *B* upon *P* and *Q*. The exact amount of this attraction is computed from the effect thus measured. In this way the force of gravity exerted by the ball *A* upon the small mass *P* becomes known.

Applying the two laws of gravity, we have the following proportion, in which *R* represents the radius of the earth, *r* the distance of *P* from the centre of the ball *A*, *Q* the quantity of matter in the earth, and *q* the quantity of matter in the ball *A*.

$$\text{Force of the Earth on } P : \text{Force of } A \text{ on } P = \frac{Q}{R^2} : \frac{q}{r^2}.$$

In this proportion the first term is the weight of *P* and is known. The second term is the gravity towards *A* found by

the method just described. The radius of the earth, the
tances r and q, and the quantity of matter in the ball A
known. The only unknown term is Q, the quantity of ma
in the earth. This may therefore be computed from the
portion. Instead of the two balls used by Cavendish, Ba
and Reich in more recent experiments have used only
with results quite as satisfactory.

354. *Second Method.* — In this method the gravity of
plummet towards a mountain (fig. 169) is found by mea
ing the deviation of the plummet from the vertical posi
when suspended near the mountain. To do this, the alti
of a star as measured by the plummet quadrant, is comp
with the altitude otherwise known; the difference wil
the angle of deviation due to the local attraction.

In the observations at Mt. Schehallion, in Scotland,
deviation towards the mountain on each side was abou
seconds. From the deviation thus found it is easy to c
pute the attraction exerted by the mountain on the plum
as compared with that exerted by the earth on the same
in other words, with the weight of the plummet. By a c
ful measurement the form and dimensions of the moun
are determined. The density of rock composing it is
ascertained. From these data we calculate first the qua
ty of matter contained in the mountain, and, second, the
sition of the point S within the mountain at which all
mass may be supposed to be collected without changing
amount of force exerted on the plummet.

The distance of this point from the plummet being den
by r, and the quantity of matter in the mountain by q, we h
the following proportion. Attraction of the earth for pl
met : attraction of mountain for plummet $= \frac{Q}{R^2} : \frac{q}{r^2}$. In t
as in the preceding case, Q, the quantity of matter in the ea
is the only unknown term, and therefore may be compu
from the proportion.

355. The results of these very different modes of obse
tion agree as nearly as could be expected with one anot
They give for the quantity of matter in the earth an amo
about equal to 5 times that of a globe of the same size c
sisting wholly of water. In other words, they show the e

to have a mean density about five times the density of water.

356. OF THE MOTION OF A BODY FALLING FROM A GREAT HEIGHT.

We have seen under a former head that the descent of a body near the earth's surface is to be regarded as a uniformly accelerated motion, because the accelerating force is supposed to be of the same intensity throughout the motion. But this supposition is inadmissible if the body fall from a great height. For from the second law of gravity we know, that the accelerating force continually increases as the body descends. Thus, at the height of 4000 miles the force is only $\frac{1}{4}$ of what it is at the earth's surface. Hence, the acceleration due to 1″ at that distance would be only $\frac{1}{4}$ of 32, that is, 8 feet, while that at the surface is 32 feet. The acceleration at a certain intermediate height would be $\frac{1}{2}$ of 32, and at a lower point $\frac{3}{4}$ of 32. Thus the motion of a body falling from the height of 4000 miles would present every rate of increment from 8 feet to 32 feet per second. As in this case the velocity would be augmented by increasing additions in successive seconds, the motion would not be a uniformly accelerated one.

357. So if the body were vertically projected to that height, its motion would be diminished by a less and less decrement for each successive second, and would not therefore be a uniformly retarded motion. But when the distance fallen through is small compared with the earth's radius, we may consider the accelerating force as constant throughout the motion, and therefore the acceleration or retardation as uniform. Even at the height of 1 mile, the acceleration is less than at the surface only by $\frac{1}{2000}$th part of 32. Hence the fall from that height would not sensibly vary from a uniformly accelerated motion.

CHAPTER XX.

OF THE ACCELERATION OF A BODY GRAVITATING ON A CURVED SURFACE, AND OF THE PENDULUM.

358. If a body or particle P be placed on the curved surface $A\ C\ B$, (fig. 170), it will descend to C with a continually increasing velocity, and passing that point, will be continually retarded in its ascent towards B. For at whatever point we suppose it to be situated for the moment, say at P_1, it is placed on an inclined plane, formed by the element of the curve at that place, the inclination of which is shown by the tangent. Hence from A to C it will be accelerated, and from C to B retarded. But as the inclination of the plane in this case varies continually, the accelerations and retardations must vary; hence the motion from A to C is not uniformly accelerated, or that from C to B uniformly retarded.

359. The rate of acceleration or value of f for different points P_1, P_2, may be deduced thus. The lines $P_1\ S$, $P_2\ S$, drawn perpendicular to the curve at P_1 and P_2, give the direction of the reactions of the curve at those points. The vertical and equal lines $P_1\ W$, $P_2\ W$, denote the constant weight P. The lines $W R_1$ $W R_2$, parallel to $P_1\ S$, $P_2\ S$, measure the reactions at P_1 and P_2, and $P_1\ R_1$, $P_2\ R_2$, the resultant accelerating force in the curve. Thus to find the accelerating force at P_1, we have $W : P_1 R_1 = g : f$, the accelerating force. So the accelerating force at P_1 : accelerating force at $P_2 = P_1\ R_1 : P_2\ R_2$.

360. It is obvious, that the direction of the force $P_3\ R_3$ being opposite to that of the motion of the body from C to B, it becomes a retarding force.

As $P\ R$ is less, the nearer the body is to C, and at that point is equal to zero, we infer that the accelerating force diminishes as P approaches the vertical, and becomes null at that point. On the opposite side of the vertical it becomes a retarding force, increasing by stages between C and B, so as to correspond in its value as a retarding force on this side, with its value as an accelerating one on the other.

361. If the points $s\ r\ B$ be taken on the ascending side of the curve, (fig. 171,) at equal heights, with $n\ m\ A$ on the descending, the velocities at these corresponding points will be equal. To prove this in the case of $m\ r$, for example, we have the velocity at $m =$ velocity at C, diminished by the acceleration received in descending from m to C; and volocity at $r =$ velocity at C, diminished by retardation caused in ascending from C to r. But acceleration in descending from m to $C =$ retardation in ascending from C to r (by 360). Hence the velocities at m and r, being less than that at C by equal quantities, must themselves be equal.

If, therefore, the body begin its motion at A, we have velocity at $A = 0$, and since velocity at $B =$ velocity at A, we have velocity at $B = 0$. That is, the body descending from A will rise to a point B on the other limb of the curve, at an equal height above C.

It follows from the above, that but for friction and the resistance of the air, a body under these conditions would continue to vibrate on the curved surface between the points A and B for an infinitely long time.

OF THE PENDULUM.

362. If we compel the particle or body to move in a curved line, not by placing it on a curved surface, as in the preceding case, but by suspending it by a cord from a fixed point O (fig. 172), the reactive tension of the cord, directed always towards O, and at right angles to the element of the curve at each point, will take the place of the reaction of the curved surface in the previous case. Thus the accelerating forces $P\,R$, $P_1\,R_1$, &c., will as before be determined by the inclination of these tangent lines to the horizon. A body thus suspended and made to vibrate is called a Pendulum.

363. Usually the curve in which a pendulum is made to vibrate is a portion of a circle. But it may also by certain arrangements be caused to vibrate in other curves, as the cycloid, &c. The entire swing of the pendulum from one side

of the vertical to the other is a *Vibration*. The time occupied in this motion is the time of the pendulum, or its *Time of Vibration*. It is obvious that the time of descending to the lowest point of the arc, as from A to C, (fig. 172), or of ascending on the other side from C to B, is one-half the time of vibration.

364. OF THE TIME OF VIBRATION OF A PENDULUM IN ARCS OF THE SAME CIRCLE, DIFFERING IN LENGTH.

Let $P\ V$ (fig. 173) be the circular arc described by a pendulum in its half vibration. Take $P_1\ V = \frac{1}{2}P\ V$ and imagine the pendulum be made also to begin its motion at P. It is proposed to compare the time of descent from P to V with the time from P_1 to V. Drawing the triangles $P\ R\ G$ and $P_1\ R_1\ G_1$ as formerly, we have, $P\ R$ and $P_1\ R_1$, the measures of the accelerating forces at P and P_1 respectively. From V let fall the perpendiculars $V\ x$ and $V\ y\ P$. These will be parallel to $P\ R$ and $P_1\ R_1$, and the triangles $V\ O\ x$ and $V\ O\ y$ will be similar to $P\ R\ G$ and $P_1 R_1\ G_1$. Hence $V\ x$ and $V y$ will be in the same proportion as $P\ R$ and $P_1\ R_1$, and therefore $V\ x$ and $V\ y$ may be taken to represent the relative intensity of the acceleration force at P and at P_1. Now $V\ y$ is equal to $\frac{1}{2}$ the chord $V\ P$. But $V\ P$ is greater than $V x$. Hence, $V\ y$ is greater than $\frac{1}{2}\ V\ x$. Hence the accelerating force at P_1 is more than half the accelerating force at P. In like manner if we assume another place for P nearer to V, and take P_1 again midway between this new place and b, we will find the accelerating force at P_1 more than half that at P. Therefore the average of the accelerating forces in the arc $P_1\ V$ is more than half the average accelerating force in $P\ V$. But the arc $P_1\ V = \frac{1}{2}$ the arc $P\ V$. If now the force urging the pendulum through $P_1\ V$ in the one case were just one-half the force urging it through $P\ V$, in the other it would describe $P\ V$ and its half $P_1\ V$ in equal times. But the force along $P_1\ V$ is more than one-half that along $P\ V$. Therefore the arc $P_1\ V$ is described in less time than its double $P\ V$. Taking other proportions between $P\ V$ and $P_1\ V$ we should obtain a similar result.

Hence, the general proposition, that *the times of vibration in arcs of the same circle differing in length, are not equal, but are greater the longer the arc.*

365. We see here a marked difference between the motions along the arcs and along their chords. For the times of the descent along the chords $P\ V$ and $P_1\ V$, and all other chords of the circle meeting at V, are by a previous proposition precisely equal.

366. When the arcs of vibration of the circular pendulum are small in comparison with the radius or length of the pendulum (fig. 174), the lines $V x$ $V y$, which represent the accelerating forces, are very nearly proportional to the arcs $P\ V$, $P\ V_1$. Hence these are described in very nearly equal times. In other words, *the vibration of a pendulum, in very small but unequal circular arcs, may practically be regarded as isochronous, that is, as performed in equal times.*

367. OF THE TIMES OF VIBRATION OF A CYCLOIDAL PENDULUM IN ARCS OF UNEQUAL LENGTH.

Let $P\ V$ (fig. 175) be the arc of a cycloid in which a pendulum is made to vibrate, and let $P_1\ V = \frac{1}{2} P\ V$. Constructing the triangles $P\ G\ R$ and $P_1\ G_1\ R_1$ as before, we have $P\ R$ and $P_1\ R_1$ for the measures of the accelerating forces at P and P_1. By the nature of the curve, $P\ R : P_1\ R_1 = P\ V : P_1\ V$. That is, in the present case, the accelerating force $P\ R$ is double the accelerating force $P_1\ R_1$. Hence, in general, the accelerating force is exactly proportional to the length of the arc to be described. Hence the average force in $P\ V$ is double the average force in $P_1\ V$, and therefore since the forces are thus proportional to the spaces through which they urge the pendulum, we have the time down $P\ V =$ the time down $P_1\ V$. The same thing may be proved for other arcs, and therefore the times of descent from all points of the cycloid to V are equal. Hence *the vibrations of a cycloidal pendulum are isochronous, or, whether great or small, are all performed in equal times.* See fig 1

368. In order to cause a suspended body to vibrate in a cycloidal curve, we hang the line of suspension between two semi-cycloids, placed in proper juxta-position. The unwinding of the cord from one curve, and its winding upon the other, will cause its end to describe a cycloid. This curious property is deduced from the geometrical nature of the cycloidal curve, (fig. 176.)

369. If P (fig. 177), be any point situated above V, but not

in the same vertical line, and if between these points we draw the cycloidal arc $P\ V$, the body by its gravity will descend from P to V along the cycloid, in less time than along any other curve line or straight line or combination of these which can be drawn between the two points. In other words, *the cycloid is the line of quickest descent.*

370. In the preceding inquiries the matter of the pendulum has been regarded as all collected at a single point P, which describes the circular or other arc, as the pendulum vibrates. Such a pendulum is entirely imaginary, and is called a *Simple or Mathematical Pendulum.* In practice, the nearest approach to this, is in a pendulum consisting of a small ball of very dense matter, such as platinum, suspended by a very slender line.

371. In the actual *Physical Pendulum*, whatever may be its form, the matter is continuously but variously distributed along the whole length. We shall presently see that the manner of this distribution has an important influence on the time of vibration of the pendulum.

372. Relation between the Time of Vibration of a simple Pendulum and its Length.

On comparing the vibrations of a ball suspended successively by a long and a short thread, we at once discover that the long pendulum vibrates more slowly than the other, that is, takes a longer time for each vibration.

The law which connects the length with the time of vibration of the *Simple Pendulum*, may be determined in two ways. First, by experiments with physical pendulums, made to approach as nearly as possible the conditions of the simple pendulum. Second, by investigating the motion of the pendulum according to the dynamical principles before proved. The former method is not mathematically precise, since the experiments cannot be made with a really simple pendulum. It is capable, however, of yielding results very near the truth.

373. *First, or Experimental Method.*

Let A and B (fig. 178), be equal small balls of lead or other dense matter, suspended by very slender threads of such length, that $S\ A$ and $S\ B$, the distances from the points of suspension to the centres of the balls, are in the ratio of 4 to 1. Let them be impelled at the same time in the same direction, so as to vibrate in the small arcs $M\,N$ and $m\,n$. It will be seen, that when A has completed one vibration, B has completed two; when A has finished two vibrations, B has finished four, and so on. Thus when $S\,A$ is four times as long as $S\,B$, its time of vibration is twice that of $S\,B$. So again, if $S\,A$ be nine times as long as $S\,B$, its time of vibration will be three times that of $S\,B$.

By extending the experiments to other lengths, we shall find the lengths and times of vibration to be related to each other according to the following series of numbers:

Length of Pendulum	$\frac{1}{9}$	$\frac{1}{4}$	1	4	9	16	25
Time of one Vibration	$\frac{1}{3}$	$\frac{1}{2}$	1	2	3	4	5

The numbers in the lower series are obviously the square roots of the corresponding ones in the upper. Hence the general law, *that the time of vibration of the simple pendulum is proportional to the square root of its length.*

374. *Second, or Deductive Method.*

Let $A\,C$ and $a\,c$ be similar arcs described by the pendulums $S\,A$ and $s\,a$, (fig. 179.) Each is a half vibration, and is performed in half the time of describing $A\ D$ and $a\,d$ respectively. We propose to compare these times, or, which is the same thing, the times of vibration of the pendulums $S\ A$ and $s\ a$. Assume $A\ M$, $a\ m$, any similar parts of $A\,C$, and $a\,c$, and take $M\ N$, $m\,n$ adjacent to these and likewise similar, but infinitely small. Draw the radii $S M$, $S\,N$, $s\ m$, $s\ n$, also the horizontal lines $M P$, $m\,p$, and the verticals $A\ P$, $a\,p$. As $M\,N$, $m\,n$ are infinitely small, they will be described in infinitely short times, and may, therefore, be regarded as described each with a uniform motion. The velocity of the pendulum along $M\,N$, is that which it has acquired by descending to M. So the velocity of the pendulum along $m\ n$ is that with which it reaches m. But since in uniform motions, $\text{Time} = \dfrac{\text{space}}{\text{velocity}}$ we have

$$\text{Time in } M\,N = \frac{M\,N}{\text{velocity at } M} \text{ and}$$

$$\text{Time in } m\,n = \frac{m\ n}{\text{velocity at } m}.$$

Let g and g_1 denote the accelerating actions of gravity at the places where the pendulums SA and sa are situated. When they are at the same spot on the earth's surface, it is obvious that g and g_1 must be equal. In order, however, to make the investigation as general as possible, we may suppose the pendulums to be at different stations, in which case g and g_1 will usually be unequal. By a former proposition, velocity at $M =$ velocity at $P = \sqrt{2g.AP}$; and in like manner, velocity at $m =$ velocity at $p = \sqrt{2g_1.ap}$.

$$\text{Hence } T \text{ in } MN = \frac{MN}{\sqrt{2g.AP}}, \text{ and } T \text{ in } mn = \frac{mn}{\sqrt{2g_1.ap}}.$$

$$\text{Therefore } T \text{ in } MN : T \text{ in } mn = \frac{MN}{\sqrt{g.AP}} : \frac{mn}{\sqrt{g_1.ap}}.$$

But from the similarity of the figures, the lines AP, ap and the lines MN, mn are in the same proportion as the radii of the circles. We may therefore substitute SA and sa for them in the above proportion, which then becomes

$$T \text{ in } MN : T \text{ in } mn = \frac{SA}{\sqrt{g.SA}} : \frac{sa}{\sqrt{g_1.sa}}.$$

$$\text{Whence, Time in } MN : \text{Time in } mn = \frac{\sqrt{SA}}{\sqrt{g}} : \frac{\sqrt{sa}}{\sqrt{g_1}}.$$

The same relation may be shown to subsist between the times of describing all the corresponding and similar elements of which AC and ac are composed. It follows, therefore, that the sums of these elementary times, that is, the entire times of describing the arcs AC and ac are in the same proportion. Hence, the whole times of vibration which are double the times in AC, ac are in this proportion; that is,

$$\text{Time of Vib}^{\text{n}} \text{ of } SA : \text{Time of Vib}^{\text{n}} \text{ of } sa = \frac{\sqrt{SA}}{\sqrt{g}} : \frac{\sqrt{sa}}{\sqrt{g_1}}.$$

It thus appears *that whatever be the relative intensity of the force of gravity accelerating the two pendulums, their times of vibration in similar arcs are proportional directly to the square root of their length, and inversely to the square root of the acceleration which measures the gravity acting on each.*

375. Although this proportion is strictly applicable only to vibrations in similar arcs, it applies without important practical error to all *small* circular arcs. For in (366) it has been shown that the times of vibration in small but unequal arcs of a given circle are practically equal. Since in using the pendulum its motion is restricted to narrow limits, we may assume all the oscillations to be equal-timed. Hence, if L and l denote the lengths, and T and t the times of vibration of two simple pendulums, we have,

$$T : t = \frac{\sqrt{L}}{\sqrt{g}} : \frac{\sqrt{l}}{\sqrt{g_1}}.$$

That is, *the time of vibration of a simple pendulum in any small circular arcs is proportional directly to the square root of the length, and inversely to the square root of the acceleration.*

376. When the accelerating force is the same for both pendulums, we have $g = g_1$, and hence $T : t = \sqrt{L} : \sqrt{l}$, which is the relation obtaining for pendulums vibrating at the same spot on the earth's surface. *Thus at any given place the time of vibration of a simple pendulum is proportional to the square root of its length.*

377. The preceding proposition enables us to compute the relative times of vibration of different pendulums, but does not afford us the means of calculating the *absolute time* of vibration of a pendulum, unless that of some other known pendulum be given.

But, since the time of vibration T is in all cases proportional to $\frac{\sqrt{L}}{g}$, it is evident that T must be equal to $\frac{\sqrt{L}}{\sqrt{g}}$ multiplied by some constant quantity. By a further investigation, not elementary enough for the plan of this work, it can be shown that the constant in question is 3.14159, &c., $= \pi$, the number which measures the circumference of a circle whose diameter is unity. The formula for determining the time of vibration of a simple pendulum, oscillating in small circular arcs, is therefore the following : $T = \pi . \frac{\sqrt{L}}{\sqrt{g}}$.

Here L denotes the length of the pendulum in feet or inches, and g the acceleration, that is, the velocity, counted in the same units, which gravity generates in 1″, at the spot where the pendulum is placed.

378. The length of a simple pendulum vibrating exactly once in a second is called the *Seconds Pendulum*. At a given spot on the earth's surface this length is invariable. In the Tower of London, according to Kater, it is 39.13908 inches, or 3.26159 feet.

If such a pendulum be carried to the top of a mountain or to the bottom of a deep mine, or be conveyed to a place further north or south, it will no longer vibrate once in a second. On the top of the mountain and at the bottom of the mine, its vibration will be slower than at the surface. For, as shown in (147 and 152), gravity diminishes as we recede from the surface in either direction. At the equator it will vibrate more slowly than at any other part of the earth's surface, and at the poles more rapidly, because, as will be shown hereafter, the force of gravity is least powerful at the equator, and augments as we recede from that line towards the poles. It follows, from this, that the seconds pendulum at the top of Mount Washington or the Peaks of Otter must be shorter than on the Atlantic coast, and that it must be longer at Boston than at Charleston in South Carolina.

379. OF THE CENTRE OF OSCILLATION.—In the *Physical Pendulum* each particle of matter may be regarded as a distinct pendulum, suspended at a certain distance from the centre of motion. As the near particles, such as x (fig. 180), tend to vibrate rapidly, while the distant ones, such as y, tend to vibrate slowly, they are compelled in their united vibration to assume a certain intermediate rate of oscillation, more rapid than that of the most distant particles, and slower than that of those near the centre. Thus the suspended mass $S\,B$ will perform its oscillations in the same time as a simple pendulum of a certain length less than $S\,B$. Suppose $S\,O$ to be the length of the equivalent simple pendulum. Then, if we conceive the whole matter of the mass $S\,B$ to be collected at the point O, and still to be suspended from S, the time of its oscillation would remain the same as in its original form. The point O thus determined is the CENTRE OF OSCILLATION.

In other words, *the centre of oscillation is that point in which, if all the mass were collected, it would continue to vibrate in the same time as in the original form of the mass.*

380. The following examples will serve to show that the place of the centre of oscillation is determined by the distribution of the matter of the vibrating body in relation to the centre of suspension.

(*a*). Let the pendulum consist of two equal bodies, A and B (fig. 181), rigidly connected. The time of vibration of the whole mass will evidently be greater than that of SB alone, and less than that of SA alone. Therefore, the length of the equivalent simple pendulum will be greater than SB, and less than SA; in other words, the centre of oscillation O will be somewhere between A and B. If as in (fig. 182) the mass A be greater than B, the equivalent pendulum will be more nearly equal to SA, and the centre of oscillation will be lower than in (fig. 181). If, on the contrary, B be increased, the point O will be in a higher position than in that form.

It is not to be inferred that when A and B are equal, the centre of oscillation is midway between their centres. By referring to (fig. 181) it will be seen that the momentum of $A = A \times m$, and the momentum of $B = B \times n$. This momentum of A acts with the leverage SA, while that of B acts with SB. Thus, the rotating force of A is measured by $SA \times A \times m$, and that of B by $SB \times B \times n$. Since $m : n = SA : SB$, we have rotating force of A : rotating force of $B = SA^2 : SB^2$. Thus A exercises more control over the vibration of the system than B, and the centre of oscillation is nearer to A than to B.

(*b*). It follows from this that the centre of oscillation of a ball suspended by a thread (fig. 183), is not at the centre of the ball, but at some point O below it. Since the deviation of O from the centre of AB (fig. 181) diminishes in proportion as the ratio of SA^2 to SB^2 approaches equality, this deviation must grow less as the line SB, by which the system BA is suspended, is made longer. If SB be imagined infinitely long, SB and SA may be assumed equal, and then O will coincide with C. On the other hand, if BA be suspended by a point at the summit of B, the centre of oscillation O will be very near to A.

381. If the oscillating mass be a cylindrical or uniform prismatical rod (fig. 184), suspended by one of its ends, it may be inferred from the above that its centre of oscillation O is situated below the middle of its length. For the rotary force of the lower half is evidently much greater than that of the upper half. It can be shown from the formula for the centre of oscillation that in this case $S\,O = \frac{2}{3}$ the length of the rod.

This may be approximately proved by suspending a small metallic sphere by a delicate thread in front of the vibrating rod, and adjusting the length of the thread so as to make the ball and rod vibrate in equal times. The distance from the point of suspension to the centre of the ball will be found to be almost exactly two thirds of the length of the rod.

382. In the preceding cases all the matter of the pendulum is supposed to be below the point of suspension. We will now consider that form of the pendulum in which part of the mass is above and part below this point. $A\,B$ (fig. 185) represents a vibrating body thus formed, in which the point of suspension S is between A and B. Suppose the pendulum to be in the attitude $A\,S\,B$, after the descending motion of A has begun. The inclination of the curve at A and B being the same, the accelerating forces acting on A and B will be parallel, and tend to produce rotation in different directions. As these forces act at right angles to $S\,A$ $S\,B$, their moments or rotating powers are as follows: Force of $A \times S\,A$, and Force of $B \times S\,B$. Hence, the *efficient accelerating force* urging the pendulum back to the vertical, is equal to the *Force of* $A \times S\,A$—*Force of* $B \times S\,B$. In other words, the efficient force which operates upon the whole mass of the pendulum is the excess of the force of the matter below the point of suspension over that of the matter above this point. In the former cases, as (fig. 182), where A and B are both beneath S, the force operating on the whole mass is the sum of the rotating forces of A and B.

383. In the present case, the motion of the pendulum is slower than if B were absent, and the pendulum were simply

the mass A at the distance SA. For in the latter condition the *undiminished force* of A would be expended in moving the mass of A alone. It follows from this, that the simple pendulum equivalent to AB is longer than SA. In other words, the centre of oscillation of the pendulum AB (fig. 186), whose point of suspension S is between A and B, is at some distance below A.

384. It is obvious that as B is carried farther from S, the moment of its force is increased, and as this is in opposition to the vibrating force, the time of vibration is increased. In other words, the centre of oscillation O is made to recede farther from S. The same effect will be produced if the mass of B be augmented while its distance SB remains the same. When A and B are equal and equidistant from S, the moving force $= o$, and so when $A \times SA = B \times SB$. In these conditions, S, the point of suspension, would be the centre of gravity of AB.

385. An ingenious use is made of the above principle in Maelzel's *metronome*, an instrument designed for beating time audibly, according to any rate of vibration the operator may desire. The principal feature of this apparatus is a short pendulum, (fig. 187) SA, having a long metallic stem Sx above the axis of suspension. On this stem is a moveable brass weight, B, which may be adjusted at any point, either near to or far from S. When the vibration is to be rapid, the weight B is slipped down; when a slow motion is desired, it is shifted towards the upper end. Thus, in a small compass, we can command vibrations as slow as those of a pendulum many feet long, and all other rates of vibration, up to those of a pendulum of only a few inches in length.

386. When the vibrating mass is of some regular geometrical form, the position of its centre of oscillation can be deduced by a rule founded on the above principles. An approximate method of finding the centre of oscillation in irregular as well as regular bodies, is as follows: Suspend a small metallic ball P (fig. 188) in front of the vibrating mass, and adjust the length of SP until it vibrates in the same time as the large mass. The distance from S to the centre of P will be the radius of oscillation of this mass AB suspended from S.

387. MUTUAL CONVERTIBILITY OF THE CENTRES OF SUSPENSION AND OSCILLATION. — If the rod $S\ A$ (fig. 189) be suspended from S, we have seen (381) that its centre of oscillation will be at O, making $S\,O = \frac{2}{3}\,S\,A$. If the rod be next suspended in a reversed position by the point O, it will be found to vibrate in the same time as before, that is, in the time of a simple pendulum $S\ O$ or $O\ S$. Hence, S is the centre of oscillation when O is the centre of suspension, and O is the centre of oscillation when S is the centre of suspension. The same result would be obtained with any other form of the pendulum. Hence the general proposition, — that *when the centre of oscillation of a pendulum is by inversion changed into a centre of suspension, the previous centre of suspension becomes the new centre of oscillation; and the time of oscillation of the pendulum is the same, by whichever of these points it may be suspended.*

388. THE TIME OF VIBRATION OF A PENDULUM IS INDEPENDENT OF THE KIND OF MATERIAL OF WHICH IT CONSISTS. — This may be proved by comparing the vibrations of balls of ivory, glass, lead, wood, &c., of equal diameter, and suspended by equal threads in a vessel exhausted of air. It will be seen that the time of vibration is the same for all. The position of the centre of oscillation being the same in all, the true length of the pendulum will be the same.

389. This fact affords another proof of the first law of gravity, viz., that the force of gravity is proportional to the quantity of matter in the gravitating body. For in order that the times of vibration may be equal, the accelerating forces must be proportional to the quantities of matter moved. But these accelerating forces are proportional to the whole gravities of the balls. Therefore, the forces of gravity acting on the different balls, are in proportion to their quantities of matter.

390. OF THE LENGTH OF THE SECONDS PENDULUM. — To find by observation the length of the seconds pendulum at a given place, two methods are used. First, by the variable pendulum; second, by the absolute or invariable pendulum.

First. By the variable pendulum. —This consists of a rod of metal (fig. 190) provided with the knife edges facing one another, and capable, by a sliding motion, of being adjusted at any positions on the rod. This pendulum, suspended by one of its knife edges, on a horizontal plate of steel or agate, in front of the pendulum of an astronomical clock, is set in vibration. The number of vibrations it performs in a given time, say 30 minutes, is found by counting how often in this time its swing coincides with that of the clock pendulum, and marking which it is that gains upon the other. Thus, suppose the number of coincidences to be 10 in the 30 minutes, and that the vibration of the clock pendulum is the faster of the two, then for each coincidence, the clock has made one vibration more than the knife-edge pendulum. Hence in all the 30 minutes the former has made 10 more vibrations than the latter. But the whole number of vibrations of the clock pendulum is $30 \times 60 = 1800$. Hence, the whole number of vibrations of the knife-edge pendulum is $1800 - 10 = 1790$; that is, the knife-edge pendulum performs 1790 vibrations in 1800 seconds, or one vibration in $\frac{1800}{1790}$ths of a second. In this way the exact time of vibration of the experimental pendulum is found.

Let the pendulum be now vibrated in the *inverted position*, and by successive trials let the positions of the two knife edges be so adjusted as to make the time of vibration the same in both attitudes of the pendulum. Measuring the distance between the two knife edges, we have, by (387), the length of the equivalent simple pendulum. From this we compute the length of the seconds pendulum at the same place, by the following proportion. The time of vibration of the experimental pendulum, say $\frac{1800}{1790}''$ is to $1''$ as the square root of the distance between the knife edges is to the square root of the length of the seconds pendulum.

391. *Second. By the absolute or invariable pendulum.*— This is a small ball of platinum, suspended by a wire of the same material, from a knife edge which rests on a plane of agate (fig. 191). Knowing the diameter of the ball, and the length and diameter of the wire, it is easy by calculation to find the exact position of the centre of oscillation of the pendulum. Measuring from the centre of suspension to this point, we have the true length of the pendulum, and this is invariable as long as the dimensions of the wire and ball

remain unaltered. To find from this the length of the seconds pendulum, we first find the time of vibration of the absolute pendulum by the method of coincidences, as above, and then use the following proportion. Time of vibration of absolute pendulum is to 1″ as square root of length of absolute pendulum is to square root of length of seconds pendulum.

The variable convertible pendulum was first used by Kater; the absolute pendulum, above described, by Borda.

392. TO COMPARE THE FORCE OF GRAVITY AT DIFFERENT POINTS ON THE EARTH'S SURFACE. — This is done by finding experimentally the length of the seconds pendulum at each place, using either the variable or absolute pendulum (390, 391.) Observation shows that the seconds pendulum is longest at the poles and shortest at the equator. But we have seen that the longer pendulum requires the greater force to make it vibrate in the same time with one that is shorter. We infer, therefore, that the *force of gravity is greatest at the poles, and least at the equator.*

393. The exact law, according to which this force varies with the change of position on the earth's surface, may be thus deduced.

Let l and l_1 be the lengths of the second pendulum at two places as determined by experiment, and let g and g_1 denote the accelerations, or velocities generated by gravity in one second at those places respectively. The forces of gravity at the two points will evidently have the proportion of g to g_1 because these are their respective effects in equal times. It has been proved (375) that the times of vibration of any two pendulums thus placed will be in the ratio of $\frac{\sqrt{l}}{\sqrt{g}}$ to $\frac{\sqrt{l_1}}{\sqrt{g_1}}$. But in the present case the time of vibration is the same, namely 1″, for both the pendulums. Hence we have $\frac{\sqrt{l}}{\sqrt{g}} = \frac{\sqrt{l_1}}{\sqrt{g_1}}$ and therefore $\frac{l}{g} = \frac{l_1}{g_1}$. From this we deduce the proportion

$$g : g_1 = l : l_1.$$

That is, *the force of gravity at different places on the* *earth's surface is proportional to the length of the seconds pendulum at those places.*

Since by observation we find that a change in the length of the seconds pendulum accompanies every change of latitude, we infer a corresponding change in the intensity of the force of gravity. Instead, therefore, of regarding g as a perfectly constant quantity, it is necessary for exact calculations, to compute its value for each particular latitude.

To effect this, we must first ascertain by experiment the length of the seconds pendulum at the given place. Let this be denoted by l, and let g represent the acceleration, or force of gravity which is to be computed; we have by (377)

$$\text{Time of vibration or } 1'' = 3.14159 \times \frac{\sqrt{l}}{\sqrt{g}}.$$

$$\text{Therefore, } \sqrt{g} = 3.14159 \times \sqrt{l}.$$

$$\text{And } g = (3.14159)^2 \times l\,;\ \text{ or } g = 9.86 \times l.$$

That is, *to compute the value of the acceleration g proper to any place on the earth's surface, we multiply the length of the seconds pendulum at that place by the number* 9.86.

394. The variation of gravity accompanying a change of latitude is caused in part by the different distances of the different parallels (fig. 192) from the earth's centre, that is, by the spheroidal form of the earth. Thus, places in the parallel of latitude *A B* are nearer the centre than those in the equator *Q Q*. Hence, gravity is stronger at *A B* than at *Q Q*. But, this force is also modified by the daily rotation of the earth around *P P*. The variation of g or of l the seconds pendulum, after allowing for the effect of rotation, may be regarded as entirely due to the want of sphericity of the earth, and furnishes the means of computing the distances of different points of the surface from the centre, that is, determining the figure of the earth as a spheroid.

395. USES OF THE PENDULUM.

1st. To determine the local value of the acceleration g, which enters into various formulæ for accelerated and retarded motions.

2d. To aid in determining the figure of the earth as above explained.

3d. To maintain uniformity in the motion of clock work, and thus to furnish a convenient and accurate keeper and teller of time.

4th. To furnish an invariable basis or standard for a system of measures and weights.

The two first of these heads have already been considered, the third will be taken up hereafter, and we now proceed to the fourth.

396. OF THE PENDULUM AS AN INVARIABLE BASIS FOR A SYSTEM OF MEASURES AND WEIGHTS.

We have seen above, that the length of the seconds pendulum at a given place on the earth's surface is invariable, as long as the force of gravity at the place continues unchanged. If then the inch be assumed as a certain part of this length, the length of the inch and of all the measures referred to it as the unit becomes invariable. Hence, whatever errors may gradually arise in the measures used in the country, the original standard can readily be restored by recurring to the seconds pendulum. The relation of the inch to the pendulum is fixed by legal enactment, declaring, in England, for example, that the seconds pendulum in the Tower of London is 39.13908 inches in length.

The inch being thus invariable, so must be the foot, yard, mile, and other measures of length. So also must be the square foot, square mile, &c. From these we directly derive all cubic measures and all systems of weights.

397. In order to make this standard perfectly uniform and precise, all the physical conditions which can in any degree affect the vibration of the pendulum must be exactly noted. Unless these can be reproduced in future trials, or can be properly allowed for, it is obvious that the length of the standard thus determined at distant intervals will not be invariable. The principal conditions to be marked, are the Temperature and Density of the air in which the experiment is made, and the Position and Mass of the surrounding walls and of the buildings in the immediate vicinity. Any great change in either of these will cause a sensible alteration in the rate of vibration of the pendulum, and therefore in the length of the seconds or standard pendulum.

OF CURVE-LINE MOTION.

CHAPTER XXI.

GENESIS OF CURVE-LINE MOTION. CENTRIPETAL FORCE.

398. As a curve is in reality but a succession of infinitely short right lines, so every curve-line motion is a succession of right-line motions, each through an infinitely short space, and occupying an infinitely short time.

399. Analysis of curve-line motion. — Since the successive elements of the curve (fig. 193) coincide with tangents at P_1 P_2 &c., it is obvious that the momentary direction of the motion at these points coincides with the direction of these tangents. If the body move in a polygon of finite sides, (fig. 194), we see that there must be a change in the direction of the motion whenever the body passes from one side of the figure to the next adjoining side, and there must, therefore, be as many such changes in a given time as there are angles passed through by the moving point or mass.

This change in the direction of the motion implies that the body is acted upon by a changing cause or force at each angle of its path. If the body moving in the direction *A B* attain the point *B*, and if no new force be supposed to act upon it, we know that by the law of inertia it will continue its motion in the line *A B T*. But, as on reaching *B*, we observe the mass suddenly to depart from the previous line of motion, and to move in the new direction *B C*, we infer that some force in a transverse direction has acted upon it, and that this force, compounded with that of its original motion in the direction $B\ T$, causes the actual motion in $B\ C$.

Assuming $B\ S\quad C\ S$, &c. to be the true directions of the transverse forces acting at B and C, and supposing the sides $B\ C\quad C\ D$, &c. to be all described in equal times, we may ascertain the comparative magnitude of the transverse forces by the process of resolution of motions. On $B\ C$ construct the parallelogram $B\,a\ C\,f_1$. Since the motion $B\,C$ is the resultant of the motions $B\,a$ and $B\,f_1$, it is obvious that $B\,f_1$ represents the motion impressed by the transverse force acting in the direction $B\ S$ during the time in which the body moves from B to C. So, by constructing the parallelogram $C\,b\ D\,f_2$, we have $C\,f_2$ for the motion impressed by the transverse force acting at C during an equal time. The transverse forces will thus be proportional to $B\,f_1$ $C\,f_2$ &c.

Since in any curve line the points $B\ C\ D$ and E are infinitely close to each other and infinitely numerous, the direction of the motion must be continually changing. Therefore, the transverse force $B\,f$ does not act at intervals, but acts incessantly. This force is usually called the DEFLECTING FORCE. The force in the direction $B\ T$, or in the direction of the tangent $P\ T$, is called the PROJECTILE OR TANGENTIAL FORCE.

We thus see that *all curve-line motion results from the combined action of projectile and deflecting forces, and is but a continued composition of the two in the successive points of the curve.*

400. SYNTHESIS OF CURVE-LINE MOTION. — In the preceding inquiry we have assumed the polygonal or curve-line motion, as already known, and by resolution or analysis have ascertained the forces which combine in its production. We now reverse the procedure, and, assuming the projectile and deflecting forces as known, we propose to show that by combining them curve-line motion is produced.

Suppose a body or point to be impelled in free space in the line $A\ T$ (fig. 195), with a velocity which in one second would carry it uniformly from A to B. After reaching B, if not acted on by any new force, it would continue to move in the same direction, and in the next second would describe the space $B\ M$ equal to $A\ B$. But imagine at B an impulse to act upon it in the direction $B\ S$, and of such intensity as in 1″ to carry it through $B\ f_1$. The body will describe the diagonal of the two impressed motions, that is, $B\ C$, uniformly in the second second. In the third second, if not acted on by any new force, it would continue in the same direction, and describe $C\ N$ equal to $B\ C$. But at C let an impulse again operate in the direction $C\ S$, and let $C\ f_2$ be the motion in 1″ due to this force. The body will describe the diagonal $C\ D$ in the third second. Thus, supposing the deflexions or impulses to follow each other at the interval of a second, the body will be seen to describe a polygonal line having as many sides as there are seconds in the time elapsed. Imagining the centripetal actions to be in more quick succession, say 100 in 1″, the number of diagonals described in the given time will be one hundred-fold, and the length of each space correspondingly less. Thus, when the centripetal force is incessant, the number of sides becomes infinite, and the polygon becomes a curve.

401. The deflecting force at different points of the curve may have the same or different intensities, may be in parallel directions, or may be in lines mutually convergent or divergent. When its directions all meet at a point, this point is called the CENTRE OF FORCE, and the deflecting force is usually termed the CENTRIPETAL FORCE.

We have a familiar example of centripetal force in the action of gravity upon a stone projected in a horizontal or oblique direction. In this case the motion of the stone is continually deflected by a force acting towards the earth's centre, and it is thus compelled to move in a curve line (figs. 196), of which we shall hereafter treat.

402. When a ball at the end of a cord or rod is made to revolve in a circle, the molecular force of the cord which connects the ball with the centre, is a true centripetal force. For if the cord be severed when the ball is at any point of its path, as at P, (fig. 197), the projectile force or inertia of the ball now acts alone, and the ball moves in the tangent $P\ T$ with the velocity it had on reaching P. The centripetal action, by which the cord incessantly deflects the motion from the projectile or tangential direction, and keeps the

ball to the curved path, is not, as sometimes alleged, the ordinary cohesion of the cord. This is proved by the fact that if the ball rest on a horizontal table, in the plane of the figure (fig. 198), the cord *C A* exerts no force upon it as long as the ball is not moving. If now the ball be impelled by a force acting in the direction *A T*, it will for the moment move in that direction, and will thus stretch the cord, and it will continue to stretch it more and more, until the molecular attraction excited in the cord is sufficient to prevent the ball from receding any further from *C*. It is in this stretched condition only that the cord can exert a centripetal force, and it is obvious that ***in such cases the centripetal force is nothing more than the tension maintained in the string.***

403. The intensity of the centripetal force excited by the motion of the ball, might be measured practically by interposing somewhere in *C A* a spring balance, as in (fig. 199), and noting the position of the index when *A* is steadily revolving with a given velocity. This would give in pounds or ounces the measure of the centripetal force. On increasing the velocity of revolution, the cord and spring would be found to suffer farther extension, thus showing that a greater centripetal force is necessary to retain the body in its path when the velocity of its motion and therefore its momentum is greater. There is evidently, therefore, a certain relation between the velocity or momentum of the revolving body that is its projectile force, and the centripetal force necessary to the retention of the body in the given circle of revolution. Increase the projectile velocity, and if you do not at the same time increase the centripetal force, the body will depart from its previous path, *P Q* (fig. 200), into an outer one *P m*. Increase the centripetal force without a corresponding change in the projectile force, and the body will leave the path *P Q* for an inner one *P n*.

404. OF CENTRIFUGAL FORCE, AND ITS RELATION TO CENTRIPETAL. — In this balanced action between the centripetal and projectile forces, it is evident that the centripetal force just counteracts that part of the projectile force which acts in a direction away from the centre. To illustrate this geometrically, let *P Q* (fig. 201) be an element of the curve, that is, an infinitely short portion of the curve. It will coincide

with the chord, and will be the diagonal of the parallelogram *P R Q K*, in which *P R* represents the projectile, and *P K* the centripetal force acting at *P*. On *P R* construct the parallelogram *P Q R L*. The projectile force is thus resolved into *P Q* and *P L*, of which *P L* represents that part of the projectile force which tends directly from the centre. But *P K* is equal to *Q R*, and *Q R* is equal to *P L*. Therefore, *the centripetal force at any point of the curve is always equal to that part of the projectile force which tends to carry the body directly away from the centre.*

This centrifugal element of the projectile force is what is called the CENTRIFUGAL FORCE.

It thus appears that whenever a body moves in a curve, *the centripetal force at each point of the path is equal and opposite to the centrifugal force.* It must be borne in mind that the centrifugal force is not a third force, additional to the projectile and centripetal, but is a part only of the projectile force. Curve-line motion does not arise from the combination of these three forces, but from that of the projectile and centripetal forces, the latter merely balancing the centrifugal element of the former, and thus leaving as a residuary motion, that in the element of the curve.

405. When a body is revolving in free space, having no mechanical connection with the centre of force, the centripetal force is manifested simply by the curve-line motion. It is thus, for example, with a projectile, or with one of the planets. But when the revolving body is mechanically connected with the centre by a rod or cord, as in (402), the centripetal force is not only evinced by the continual deflection which keeps the body in a curved path, but is shown by the tension of the cord, and may, as we have seen, be measured on the scale of the spring-balance. Here the cord or rod is in the same condition as if it were at rest, and were stretched to the degree of tension, which is caused by the revolving motion. The stretching force acting from the centre outwards, and the molecular attraction acting towards the centre, being in equilibrium, are equal and opposite to one another. The stretching force is identical in quantity and direction with the centrifugal force, and the molecular attraction with the centripetal force.

406. We have already seen how this stretched state of the cord is caused in the outset of the motion (402) by the projectile force actually carrying the body farther and farther from the centre until the centrifugal element of this force is just balanced by the tension of the cord. It remains now to explain in what way the same tension is maintained during the whole of the circular motion. Supposing the velocity to be uniform, the centrifugal force must, as we have seen, (403) bear a certain ratio to the projectile force at each point of the path. In the present case, the velocity being uniform, the projectile force is the same at all points of the curve.

Hence its centrifugal element, that is, the centrifugal force, must be every where the same. But this is the force which stretches the cord or rod. Therefore, the extension of the cord must remain the same in all the successive positions of the revolving mass, and hence the molecular reaction of the cord, that is, the centripetal force, must continue unchanged.

Thus the uniform motion of the revolving body maintains equality in the centrifugal element at the successive points of the curve; this keeps the cord equally stretched, and from this uniform tension arises a uniform centripetal force.

407. The projectile force, in other words, the force of the revolving body due to its motion at any point of its path, is evidently dependent upon the velocity and the quantity of matter. Increase either of these, and in the same ratio you will augment this force. In a word, *the projectile force at each point of the curve is the momentum of the revolving mass while passing that point*, and is measured by $Q\ V$, that is, the quantity of matter multiplied by the velocity of the mass. Thus a mass 10 revolving with the velocity 20, and a mass 8 with the velocity 12, will have projectile forces in the proportion of 10×20 to 8×12, that is, 200 to 96.

408. Definitions. — When a body revolves in free space around a given centre of force in a circular, elliptical or other path, we may conceive a line connecting the revolving body with the centre to move with the body, shortening or lengthening itself according as the curve approaches or recedes from the centre of force. This line is called the ***Radius Vector.*** The curved path of the body is usually called the ***Orbit*** or Trajectory.

409. **RELATION OF THE AREAS PASSED OVER BY THE RADIUS VECTOR, TO THE TIMES IN WHICH THEY ARE DESCRIBED.**

When the body revolves in a circle with a uniform velocity, the centre of force being at the centre of the circle, the radius vector is the radius of the circle. The areas $S\,P\,Q$ $S\,P_1\,Q_1$ (fig. 202) described by the radius vector are proportional to the arcs $P\,Q$ $P_1\,Q_1$, that is, $S\,P\,Q : S\,P_1\,Q_1 = P\,Q : P_1\,Q_1$. But from the uniformity of the orbital motion we have $P\,Q : P_1\,Q_1 =$ time in $P\,Q$: time in $P_1\,Q_1$. Therefore, $S\,P\,Q$ is to $S\,P_1\,Q_1$ as the time of describing $S\,P\,Q$ to the time of describing $S\,P_1\,Q_1$. That is, in this the simplest case of revolving motion, *the areas decribed by the radius vector are proportional to the times.*

410. The proportionality of the areas described by the radius vector to the times in which they are described, is true of every form of orbit and of every position of the centre of force in regard to the orbit.

To prove this proposition, we must recur to the genesis of curve-line motion as explained in article (400). Let $A\,B\,C\,D$ (fig. 203) be a polygonal path described around the centre of force, S, the sides being described with a uniform motion due to the combined action of the projectile and deflecting forces at $B\,C\,D$, &c. By the construction explained in (400) we have $B\,a = A\,B$ and $C\,b = B\,C$. Draw the radii $S\,A$ $S\,B$ $S\,C$ and join $S\,a$.

The triangles $S\,a\,B$ and $S\,B\,C$ are equal, for they have the same base $S\,B$ and stand between the same parallel, $S\,B$ and $C\,a$. Also the triangles $S\,a\,B$ and $S\,A\,B$ are equal, because they have equal bases $A\,B$, $B\,a$, and a common vertex S. Therefore, the triangles $S\,B\,C$ and $S\,A\,B$ are equal. In the same manner we may prove that the triangle $S\,C\,D$ is equal to the triangle $S\,B\,C$. Thus the triangles $S\,A\,B$, $S\,B\,C$, $S\,C\,D$, &c. are all equal. It follows that all the triangular spaces corresponding to the successive seconds of the polygonal motion are equal. The same must evidently be true for the triangles corresponding to the infinitely short sides of the polygon, which are described in equal times when the polygon becomes a curve. Therefore, when the motion is performed in a curve, the radius vector passes over equal triangular spaces in equal times, and hence in a double, triple, &c. time describes a double, triple, &c. area.

Thus, therefore, it appears that ***whatever be the form of the orbit in which a body revolves around a centre of force, whatever the position of that centre, the motion is always such that the radius vector describes areas proportional to the times.*** This proposition forms the basis of nearly all investigations relating to orbital motions.

411. The converse proposition is also true, viz., that ***if a point be found within any orbit, having the property that the radius connecting it with the revolving body describes areas which are proportional to the times, that point is the centre of force.*** The demonstration is as follows.

Let $A B$ $B C$ $C D$, (fig. 204,) be elements of the curve which are described in equal and infinitely short times, and let S be a point so situated that the areas $S A B$ $S B C$ $S C D$, &c., are equal. Prolong $A B$ indefinitely and take $B M$ equal to $A B$. It is obvious that the deflecting force acts in succession at the points B C D, &c. The line $B M$ denotes the space which would be described in the second element of time, supposing no deflexion to take place. But $B C$ is the space actually described in that time, through the combination of the projectile motion $B M$ and the deflexion acting at B. Join $M C$ and complete the parallelogram $B M C x$. It is obvious that $B x$ represents the deflexion which acts at the point B. We can now prove that $B x$ lies in the line $B S$. Join $M S$. By hypothesis the triangles $S A B$ and $S B C$ are equal. But the triangles $S A B$ and $S B M$ are equal because they have equal bases $A B$ and $B M$, and a common vertex S. Therefore the triangle $S B C$ is equal to the triangle $S B M$. But these triangles stand on a common base $B S$, therefore they must lie between the same parallel lines, and, hence, $M C$ is parallel to $B S$. But by construction, $B x$ is parallel to $M C$; therefore, $B x$ lies in the line $B S$. But we have seen above, that $B x$ denotes the deflexion at B. It follows, therefore, that the deflecting force which acts on the body at B is directed towards S. In like manner may it be proved that the deflecting force at C D and all other points of the curve is directed towards S. Thus it appears, that when the point S is so situated that equal areas are described around it in equal times, this point is the centre of force.

412. The principle of equal areas just proved, affords an

easy means of comparing the velocity of a revolving body at different parts of its orbit.

Thus, if a body revolve in a circle, and the centre of force, instead of being at the centre of the circle, is at some other point, as S (fig. 205), its velocity will not be uniform, but will be greater as the body is nearer to S. Let $S P Q$ and $S P_1 Q_1$ be equal areas. They will correspond to equal times. Hence $P Q$ and $P_1 Q_1$ must be described in equal times; but since $P_1 Q_1$ is greater than $P Q$, the velocity in $P_1 Q_1$ is greater than in $P Q$. So when the body describes an ellipse, the centre of force being at S, (fig. 206), the velocity in $P_1 Q_1$ is greater than in $P Q$.

413. To determine the precise relations of the velocities at different points, P and P_1, of any orbit, (fig. 207), let v and v_1 be the velocities at P and P_1 and $P Q$ $P_1 Q_1$, infinitely short arcs described in equal times. As these times are infinitely short, $P Q$ and $P_1 Q_1$ will be described respectively with the velocities v and v_1 unchanged. Hence $P Q : P_1 Q_1 = v : v_1$. But since the times are equal, the area $S P Q$ is equal to the area $S P_1 Q_1$ (410). Drawing $S t$ $S t_1$ at right angles to the tangents $P t$ $P t_1$, we have $S P Q = \frac{1}{2} P Q \times S t$ and $S P_1 Q_1 = \frac{1}{2} P_1 Q_1 \times S t_1$. Therefore $P Q \times S t = P_1 Q_1 \times S t_1$. Hence $P Q : P_1 Q_1 = S t : S t_1$ or $v : v_1 = S t : S t_1$. Hence it follows, in general, that *the velocities at different points in an orbit are inversely proportional to the perpendiculars let fall on the tangents from the centre of force.*

414. It follows from this, that in the orbit (207) the velocity continually grows less as the body moves from A towards B. When the curve returns into itself, as in (figs. 205, 206) there are points, as A and B, where the radius vector is perpendicular to the element of the curve. Hence the velocity at A is to velocity at B as $S B$ to $S A$. Since in the ellipse (fig. 206) $S A$ is the shortest of all the perpendiculars, the velocity at A is greatest, and since $S B$ is the longest perpendicular, the velocity at B is the least.

415. MEASURE OF THE INTENSITY OF THE CENTRIPETAL FORCE AT DIFFERENT POINTS OF THE SAME ORBIT AND IN DIFFERENT ORBITS.

To compare the centripetal force as it varies *from point to point in the path of a revolving body*, we assume arcs

$P\ Q$, $P_1\ Q_1$, (fig. 208), described in equal but infinitely short times. We then draw the tangents $P\,t$, $P_1\ t_1$, and the lines $Q\ R$ $Q_1\,R_1$ parallel respectively to $S\,P$ $S\,P_1$. Since the angles at S subtended by $P\ Q$ $P_1\,Q_1$ are infinitely small, we may consider $S\,Q$ as parallel to $S\,P$, and $S\,Q_1$ to $S\,P_1$. Hence, while the body moves from P to Q, the actions of the centripetal force upon it are to be regarded as parallel to $S\,Q$; and so also these actions during the motion from P_1 to Q_1 are parallel to $S\,Q_1$. Thus during the time of describing $P\ Q$, the body has been drawn from the tangent to the point Q by a force always parallel to $S\,P$. Therefore $R\ Q$ represents the whole deflexion in the given element of time, and so $R_1\ Q$ will represent the deflexion corresponding to $P_1\,Q_1$. But since any force, however variable in its intensity as compared at finite intervals, is constant in its action during an infinitely short time, we infer that the centripetal force is of constant intensity, while impelling the body through $R\ Q$, and again of constant intensity through $R_1\ Q_1$. Thus $R\ Q$ and $R_1\ Q_1$ are the spaces through which the centripetal forces impel the body by their uniform action in equal elementary times. The forces are therefore proportional to these deflections, that is, Force at P : Force at $P_1 = R\ Q : R_1\ Q_1$.

416. The same theorem enables us to compare the centripetal forces *in different orbits.* Thus let $A\ B$ $A_1\,B_1$ be the curves described about the two distinct centres of force $S\ S_1$ (fig. 209), and let it be required to compare the centripetal force at P in the one orbit with that at P_1 in the other. We assume, as before, the arcs $P\ Q$ $P_1\,Q_1$ described in equal but infinitely short times, and complete the figure as above. By the reasoning used in the preceding demonstration, it is obvious that $R\,Q$ and $R_1\ Q_1$ are the spaces through which the centripetal forces impel the two bodies in equal infinitely short times by their uniform action. Therefore Force at P : Force at $P_1 = R\ Q : R_1\ Q_1$.

Thus is established the fundamental theorem, *that the central force at different points of any orbit or orbits are proportional to the lines of deflexion corresponding to arcs described in equal infinitely short times, these lines of deflexion being parallel to the radii vectores for the given points.*

417. It follows from the above, that, to ascertain the rate according to which the centripetal force varies from point to point in any orbit, that is, to find *the law of the force*, it is only necessary to know the relative values of the deflexion RQ for the different parts of the orbit. Now the rate of variation of RQ from point to point in any curve, is dependent on two conditions, viz.: the form of the curve, and the position of the centre of force within it.

418. Thus when the orbit is a circle, and the centre of force at the centre of the circle, the value of RQ is the same for every part of the orbit. For we have already seen that the velocity of revolution is then uniform. Hence the elementary arcs PQ P_1Q_1 described in equal times are equal, and therefore the deflexions RQ R_1Q_1 are equal (fig. 210). But when the centre of force is not at the centre of the circle, it is obvious that RQ varies from point to point. Thus in (fig. 211), if S be the centre of force, and if we take the arcs PQ P_1Q_1 as described in equal infinitely short times, these arcs must be such as to make the areas SPQ $S_1P_1Q_1$, described by the radius vector in those times equal. Hence PQ P_1Q_1 cannot be equal, and therefore RQ must vary.

So, again, when the orbit is an ellipse, the variations of RQ follow one law when the centre of force is at the centre of the ellipse, and a very different law when it is at the focus of the ellipse.

419. It follows, from what has been said, that knowing *the form of an orbit and the place of the centre of force within it*, we can always deduce the law of variation of the deflexion from point to point, and thus determine *the law of the centripetal force* in the given orbit.

The following are among the most interesting examples of the application of this principle in demonstrating the law of centripetal force.

420. WHEN THE ORBIT DESCRIBED BY THE REVOLVING BODY IS A CIRCLE, AND THE CENTRE OF FORCE IS AT THE CENTRE OF THE CIRCLE, TO FIND THE LAW OF THE FORCE.

We have before proved (418), that in this case the deflexion RQ is the same for all points of the orbit, the centripetal force must therefore be equal for all points. This is the law of force when the orbit is a circle, and the centre of the force is at the centre of the circle.

421. WHEN THE ORBIT DESCRIBED BY THE REVOLVING BODY IS AN ELLIPSE, AND THE CENTRE OF FORCE IS AT THE CENTRE OF THE ELLIPSE, TO FIND THE LAW OF THE FORCE.

Assuming the sectoral areas SPQ SP_1Q_1 (fig. 212) to be equal and indefinitely small, it follows from the properties of the ellipse that $RQ : R_1Q_1 = SP : SP_1$. But the areas SPQ, SP_1Q_1 being equal, must be described in equal times. Hence, RQ, R_1Q_1 correspond to equal infinitely short times, and are proportional to the centripetal force at P and P_1 respectively. Therefore, force at P : force at $P_1 = SP : SP_1$. That is, *when the centre of force is at the centre of the ellipse, the intensity of the force is proportional to the distance of the revolving body from the centre of the curve.*

422. WHEN THE ORBIT IS AN ELLIPSE AND THE CENTRE OF FORCE IS AT ONE OF THE FOCI OF THE CURVE, TO FIND THE LAW OF THE FORCE.

Taking S as the focus of the curve, (fig. 213), and assuming the two sectoral areas SPQ and SP_1Q_1 equal and infinitely small, it is deducible from the nature of the curve that $RQ : R_1Q_1 = SP_1^2 : SP^2$. But by the same reasoning as above (421), the lines RQ R_1Q_1 are in the proportion of the forces at P and P_1 respectively. Therefore, force at P : force at $P_1 = SP^2 : SP_1^2$. That is, *when the centre of force is at a focus of the ellipse, the centripetal force is proportional inversely to the square of the distance of the revolving body from the focus.*

The law of force just demonstrated is of great interest from its being that which regulates the gravitation of a planet at different points of its orbit.

423. WHEN THE ORBIT IS A SPIRAL, THE CENTRE OF FORCE BEING AT THE CENTRE OR POLE OF THE SPIRAL, TO FIND THE LAW OF FORCE.

Let S be the centre of the spiral, (fig. 214), and take infinitely small areas SPQ SP_1Q_1 and draw the other lines as before. From the properties of the spiral it may be proved that $RQ : R_1Q_1 = SP_1^3 : SP^3$. Therefore, force at P : force at $P_1 = SP_1^3 : SP^3$. That is, *in the spiral the* *centripetal force is proportional inversely to the cube of the distance of the revolving body from the centre of the spiral.*

424. These examples serve to prove what was but partially illustrated before (419), that in any case of motion about a centre of force, the law of the centripetal force can always be deduced from what we know of the nature of the curve and the place of the centre of force. *In a given curve and with a given place for the centre of force, there is but one law of force possible.*

425. OF THE RELATION OF THE FORCE, VELOCITY, AND RADIUS VECTOR IN CIRCULAR ORBITS, THE CENTRE OF FORCE BEING AT THE CENTRE OF THE CIRCLE.

Let $A\,B$ be a circular orbit, (fig. 215) and S the centre of force coinciding with the centre of the circle. It has been proved that the motion of revolution will be uniform, and that the centripetal force will be the same for all points of the orbit. Let f represent the acceleration which this force by its uniform action would impart in 1″. Let v denote the velocity of revolution, and t the element of time in which $P\,Q$ is described. We have $P\,Q = v\,t$. By geometry $R\,Q = \frac{P\,Q^2}{P\,D}$, therefore, $R\,Q = \frac{v^2\,t^2}{2\,r}$, r being the radius of the circle. But $R\,Q$ being due to constant force, is described with a uniformly accelerated motion in the time t. Therefore, $R\,Q = \frac{1}{2} f\,t^2$. Equating the two values of $R\,Q$ we have $\frac{1}{2}\,f\,t^2 = \frac{v^2\,t^2}{2\,r}$. Hence, $f = \frac{v^2}{r}$.

That is, *the acceleration which the centripetal force is capable of producing by its uniform action for one second, is equal to the square of the velocity with which the body revolves divided by the radius of its orbit.*

Example. — Suppose a body to revolve at the rate of 10 feet per second in a circle whose radius is 5 feet. We have, $f = \frac{10^2}{5} = \frac{100}{5} = 20$. That is to say, the centripetal force, supposing its action to be continued for 1″ in a uniform direction, and, with the same intensity which it exerts each moment on the revolving body, would impress upon this body a velocity of 20 feet per second.

426. From the value of the acceleration above found, we

deduce a simple expression for the velocity in a circular orbit in terms of the acceleration and the radius. Since $f = \frac{v^2}{r}$ we have $v^2 = f\,r$ and $v = \sqrt{f\,r}$. Suppose the body to be urged from a state of rest by the centripetal force acting uniformly, until the accumulated velocity is exactly equal to v; and let s be the space through which the body has been impelled during the acceleration. By (310) we have $v = \sqrt{2\,f\,s}$. Equating this with the preceding value of v, gives $\sqrt{2\,f\,s} = \sqrt{f\,r}$.

$$\text{Whence, } 2\,s = r \text{ and } s = \frac{r}{2}.$$

That is, *the velocity with which a body revolves uniformly in a circular orbit, is that which the centripetal force would generate by uniform action through a distance equal to half the radius of the orbit.*

427. To compare the centripetal forces in different circular orbits, we assume F F_1 to represent these forces in any two such orbits, and f f_1 the corresponding accelerations.

It has already been shown (297) that any constant forces are in the same proportion as the accelerations they impress on equal masses in the same time. Thus, $F : F_1 = f : f_1$. Now, let v and v_1 denote the velocities in the two orbits and r and r^1 their radii respectively. We have (425) $f = \frac{v^2}{r}$ and $f_1 = \frac{v_1^2}{r}$. Therefore, $F : F_1 = \frac{v^2}{r} : \frac{v_1^2}{r_1}$.

That is, *where unequal circular orbits are described with unequal velocities, the centripetal forces are proportional to the squares of the velocities of revolution, divided by the radii of the orbits.*

428. If the circles be equal, then $r = r_1$ and $F : F_1 = v^2 : v_1$. That is, *when equal circular orbits are described with unequal velocities, the centripetal forces are proportional to the squares of the velocities.* This relation equally applies where a given body is made to revolve in the same circle with different velocities. Doubling the velocity makes the centripetal force quadruple, &c.

429. If the circles be unequal, but the velocities of revolution be equal, we have $v^2 = v_1^2$ and $F : F_1 = \frac{1}{r} : \frac{1}{r_1}$, or

$r_1 : r$. That is, *when unequal circular orbits are described with equal velocities, the centripetal forces are proportional inversely to the radii.*

It is evident from (fig. 216), that, assuming PQ and P_1Q_1 as equal, $Q\,R$, the deflexion due to the force in the smaller circle, is greater as the circle is less. In other words, the less the radius of the circle, the more rapid is its incurvation or deviation from the tangent. The exact relation is given above, but its dependence upon the different curvature of the two circles is usefully seen in the figure.

430. If the unequal circles be described in equal times, we have the velocity proportional to the spaces, that is, to the circles described; but the circles are proportional to their radii, therefore the velocities are as the radii, and hence $v^2 : v_1^2 = r^2 : r_1^2$, therefore $F : F_1 = \frac{r^2}{r} : \frac{r_1^2}{r} = r : r_1$. That is, *when unequal circular orbits are described in equal times, the centripetal forces are proportional directly to the radii.*

431. A case of much importance, in its relation to the law of planetary gravitation, may here be considered. Suppose two circular orbits to be described, with such velocities that the squares of the times shall be proportional to the cubes of the radii of the circles, that is, $t^2 : t_1^2 = r^3 : r_1^3$. Since in uniform motions $time = \frac{space}{velocity}$, we have

$$t^2 = \frac{space^2}{v^2} = \frac{circle^2}{v^2}.$$

But since circles are proportional to their radii, t^2 will be proportional to $\frac{r^2}{v^2}$. Hence $t^2 : t_1^2 = \frac{r^2}{v^2} : \frac{r^2}{v_1^2}$. Substituting in the above proportion this last ratio for its equal, that of t^2 to t_1^2, we have $\frac{r^2}{v^2} : \frac{r_1^2}{v_1^2} = r^3 : r_1^3$. Whence, by multiplying extremes and means and reducing, we get $v^2 r = v_1^2 r_1$. But by (425) $v^2 = f\,r$, and therefore $v^2 r = f\,r^2$. So likewise $v_1^2 r_1 = f_1 r_1^2$. Whence $f\,r^2 = f_1 r_1^2$. Converting this equation into a proportion, we have

$$f : f_1 = r_1^2 : r^2.$$

Hence the proposition, — *that when in circular orbits, the squares of the times of revolution are proportional to the* *cubes of the radii, the centripetal forces are proportional inversely to the squares of the radii.*

CHAPTER XXII.

OF CENTRAL FORCES IN GENERAL, AND OF CENTRIFUGAL ACTION.

432. As the centrifugal and centripetal forces are in all cases equal, as well as opposite at every point of an orbit, it follows that the laws of variation which may be demonstrated in regard to the one, are equally true of the other. Hence, in treating of these laws, it has been found convenient to include both kinds of force under one name, and, for this purpose, the term CENTRAL FORCES is employed.

The theorems relating to circular orbits, which have been deduced above from the expression $f = \frac{v^2}{r}$, may be illustrated experimentally by the use of the apparatus called the Whirling Table.

ILLUSTRATIONS OF CENTRAL FORCES BY THE WHIRLING TABLE.

433. The whirling table consists of a horizontal wheel, *A B*, which, by means of endless cords, gives rotation to two vertical axes, *P* and P_1, (fig. 217), the relative rotation being determined by the size of the grooved pulleys on *P* and P_1, around which the cords severally pass. To the top of each axis is screwed a flyer, *MN*, (fig. 218). This is a narrow strip of wood, graduated equally from the middle to the end. Two smooth brass rods, supported a little above the graduated surface and parallel to one another, extend from near the middle to the end of the flyer. A weight *W*, connected with these bars so as to be capable of sliding freely along them, forms the revolving body. It usually weighs 1 oz.

A cord attached to this weight passes under a pulley at x, then ascends and passes over a pulley at y, and is bent downwards and fastened to a disc of brass, F. The pulley y is sustained by brass columns, between which the disc F can rise freely. From the centre of this disc, as well as that of W, there rises a short projecting rod of brass, intended to receive the additional perforated weights of 1 2 3 4, &c. ounces when needed. The string connecting W with F is capable of being drawn through clamps attached to W at m and n, and then fastened when the centre of W has been adjusted at the desired distance from the axis, as measured on the scale of the flyer.

Let us suppose the disc F to weigh 2 oz.; W being 1 oz.; and let the latter be adjusted to any distance from the axis, and the flyer set in motion. The centrifugal force of W stretches the string, and is balanced by the reaction of the string, as long as this stretching force does not exceed the weight at F. Increasing the velocity, the centrifugal force augments until it preponderates over F, and then W recedes from the centre, and, in doing so, raises F. At a certain stage of the increasing velocity it is obvious that the centrifugal force must be exactly equal to F. The weight of F truly measures the centrifugal force, and of course the centrifugal force of W, *at this particular rate of motion*, in the given circle.

Experiment I. — Let W and W_1 be adjusted respectively at the distances 12 and 6, and let W make twice as many revolutions as W_1, or have an absolute velocity four times as great. Load the centres so as to make $F = 16$ oz. and $F_1 = 2$ oz. By gradually increasing the velocity it will be seen that, at a certain speed, F and F_1 are raised at the same moment. It is obvious that, in the instant before their ascent, F and F_1 were just balanced by the centrifugal forces of the equal weights W and W_1. Thus the central forces in this experiment are 16 and 2 respectively, or in the ratio of 8 to 1. Calculating the relation of F to F_1, according to the theorem in (427), we have $v = 4$ and $v_1 = 1$ $r = 12$ and $r_1 = 6$. Hence $F : F_1 = \frac{v^2}{r} : \frac{v_1^2}{r_1} = \frac{4^2}{12} : \frac{1}{6} = 16 : 2 = 8 : 1$. Thus the experiment accords with the theorem.

Experiment II. — Let W and W_1 be adjusted at any equal distances from the axis, but let the velocity of W be three times that of W_1. Load the centre so as to make $F = 18$ oz. and $F_1 = 2$ oz. At a certain stage of motion F and F_1 will rise simultaneously. Here, therefore, the central forces are 18 and 2, and these are in the ratio of 9 to 1. Applying the theorem of (428), and observing that $v = 3$ and ${}_1v = 1$, we have $F : F = v^2 : v_1^2 = 9 : 1$.

Experiment III. — Let W and W_1 be adjusted at the distances 4 and 8, and let W revolve twice as often as W_1. Make $F = 4$ and $F_1 = 2$ ounces. At a certain stage of velocity F and F_1 will rise together. Here the central forces of W and W_1 are in the ratio of 2 to 1.

Applying the theorem in (429), we have $v = v_1$ $r = 4$, and $r_1 = 8$, and $F : F_1 = \frac{1}{r} : \frac{1}{r_1} = 8 : 4 = 2 : 1$.

434. COMPARISON OF THE CENTRAL FORCE OF A BODY REVOLVING UNIFORMLY IN A CIRCULAR ORBIT WITH THE WEIGHT OF THE BODY.

For this purpose we refer to the principle, that constant forces are proportional to the accelerations they produce in 1″. Denoting the central force by F and the weight by W, we have, $F : W = f : g$. Hence, $Fg = Wf$, and $F = W . \frac{f}{g}$.

That is, *the central force is equal to the product of the weight, by the ratio of the acceleration of the central force to the acceleration of gravity.*

Substituting for f its value, viz. $\frac{v^2}{r}$, we have $F = W . \frac{v^2}{gr}$.

435. From the value of F, above given, it is easy to infer the effect of a difference of mass on the centrifugal force of the revolving body. Thus, let two bodies whose weights are W and W_1 revolve in equal circles with equal velocities. Then will $F = W . \frac{v^2}{gr}$ and $F_1 = W_1 \frac{v^2}{gr}$. But $\frac{v^2}{gr}$ is common to these values, therefore, $F : F_1 = W : W_1$. That is, *the velocities and orbits of two bodies being equal, their centrifugal forces are proportional to their masses.*

This may be shown by the whirling table, thus: Make $W = 5$ oz. and $W_1 = 1$ oz. and let the central weights F and F_1 be 10 oz. and 2 oz. respectively. Adjusting

W and W_1 at equal distances from their centres of rotation, let them be turned with equal velocity. The loads 10 and 2 will rise at the same moment, showing that the five-fold weight W produces five-fold centrifugal force.

It is evident that the above formula $F = W \cdot \frac{v^2}{gr}$ enables us to compute either of the four quantities F, W, v or r, supposing the other three to be known. In the following examples as well as others in future, the length of the cord designates the whole distance from the centre of the circle to the centre of the ball.

Example I. — Suppose a ball weighing 2 ounces whirled by a cord 4 feet long in a horizontal circle with a velocity of 16 feet per second, with what force is the cord stretched?

$$\text{Here } F = 2 \cdot \frac{16^2}{32 \times 4} = \frac{512}{128} = 4 \text{ ounces.}$$

Example II. — Suppose a ball weighing 6 oz. to revolve horizontally at the end of a cord 12 feet in length, what must be its velocity in order that it may stretch the string with a force of 4 oz.?

$$\text{Here } F \text{ or } 4 = 6 \cdot \frac{v^2}{32 \times 12}. \text{ Hence } v^2 = 256 \text{ and } v = 16 \text{ ft.}$$

436. If the ball be whirled in a *vertical orbit*, the tension of the string will vary with the position of the ball in the circle. At the lowest point, this force will be equal to the sum of the centrifugal force and the weight, that is, $F + W$; at the highest point, it will be equal to the difference of these forces, that is, $F - W$. Thus, supposing the orbit in Example I. to be vertical, the tension of the cord for the lowest point is 6 oz., and for the highest point 2 oz.

437. In every such orbit a certain velocity must be attained, to enable the revolving body to pass the top of the circle without falling. The least velocity at which this becomes possible, is evidently that which creates a centrifugal force exactly equal to the weight of the body. If in a given case, therefore, we wish to compute this limit of velocity, we recur to the formula $F = W \cdot \frac{v^2}{gr}$. Whence we have $F \cdot gr = W \cdot v^2$. Making $F = W$, this gives $v^2 = gr$ and $v = \sqrt{gr}$, that is, the velocity necessary to enable the mass

to pass the top of the vertical circle is found, by taking the square root of 32 times the length of the cord.

Example. — Suppose an open glass vessel, containing water, to be whirled in a vertical circle whose radius is 2 feet. What must be its velocity, in order that the water shall not escape as it passes the top in an inverted position? (Fig. 219.) Here $v = \sqrt{32 \times 2} = 8$ feet per second.

438. Of the Conical Pendulum. — This is a ball suspended by a rod or cord from a fixed point V (fig. 220), as in the common pendulum. But, instead of vibrating in a vertical plane, the ball is compelled to revolve in a horizontal circle $A\,B$, by the conical motion of the rod or cord $V\,W$. Supposing the ball to be at O in its circular orbit, it is acted upon by its weight, the centrifugal force and the tension of the string. The centripetal force retaining it in the circle, is here the resultant force due to the combined action of the weight in the direction $O\,Q$ and the tension or traction of the cord in the line $O\,V$. As this centripetal force balances the centrifugal force, we have three forces acting at O in equilibrium, viz. the weight, the centrifugal force, and the tension. But, the lines $V\,S$, $S\,O$ and $O\,V$ being parallel respectively to these balancing forces, $O\,S\,V$ is a triangle of forces, and the centrifugal force, weight and tension are proportional to $O\,S$, $O\,V$ and $V\,S$. Therefore, $F : W = O\,S : O\,V$. But this latter ratio is the same as that of Tangent of $O\,V\,S$ to 1. Hence $F : W =$ Tangent $O\,V\,S : 1$.

Let φ denote the angle $O\,V\,S$, the inclination of the cord to the vertical. Then Tangent $\varphi = \frac{F}{W}$.

We thus see that the inclination of the cord to the vertical diminishes as we augment the weight, and increases as we increase the centrifugal force.

The latter effect is shown experimentally, by increasing the velocity of rotation, when, in virtue of the additional centrifugal force, the body will rise to a higher and wider circle, and the angle φ will be increased. (Fig. 221.)

439. In order to compute the exact value of this angle in any case, we replace F by $W . \frac{v^2}{gr}$. We then have Tan-

gent $\phi = W . \frac{v^2}{g r}$ divided by W; that is, Tangent $\phi = \frac{v^2}{g r}$. Knowing the velocity of rotation, and the radius of the circle in which the conical pendulum revolves, we may find the inclination of the cord, either by this formula, or by the following construction. Compute v^2 and $g r$, then take any horizontal line $O\ S$, and erect a perpendicular of such length that $O\ S$ and $V\ S$ may be in the ratio of v^2 to $g r$. Join $O\ V$, and the angle $O\ V\ S$ will be the angle required. (fig. 222).

440. Application of the Conical Pendulum. — The conical pendulum is used as a means of regulating the supply of steam in certain steam engines, and for other like purposes. The mechanism by which it operates is seen in (fig. 223). $A\ B$ is a vertical shaft which is connected with the engine through the pulley at A, and is therefore accelerated or retarded with the rest of the machine. Two heavy balls attached to iron rods, which are jointed into the moveable socket S, are made to revolve along with the shaft in a circle while the rods have a conical motion. Two smaller rods, $P\ Q$ and $P\ R$, moveable around joints at P, at Q and at R, sustain the large rods and balls, and allow them to recede more or less from the shaft, according to the velocity of the rotation. As they recede, they depress the socket S, as they approach, they raise it. This motion is communicated, by the rod and levers, to the valve x, in the steam-pipe which supplies the power. When there is an excess of power in action, the shaft is made to turn faster. This increases the centrifugal force of the balls, and they recede. But in doing so, they draw down the socket, and turn the steam-valve so as to diminish the supply. On the other hand, when there is a deficiency of power, the collapsing balls open the steam-valves and augment the supply.

441. Of the Centrifugal Force of vehicles on curves or turns in roads.

Common observation shows, that a vehicle in quickly turning a curve, on a level road, is liable to be overthrown outwards. In this case, the centrifugal force, acting through the centre of gravity, in the direction $O\ P$ (fig. 224), combines with the weight of the vehicle, in the direction $O\ Q$. The actual pressure of the vehicle is in the direction of the resultant $O\,R$. As long as this line intersects the road surface within $A\,B$, as at x, the vehicle will have stability depending (fig. 213) on the length of $B\,x$. But if the intersection should lie beyond B, as at y, the vehicle will be overthrown. The greater the intensity of the centrifugal force compared with the weight, the less the stability in turning the curve. As the centrifugal force of a given weight is proportional to $\frac{v^2}{r}$, it follows that the ***greater the velocity of the vehicle, and the less the radius of the curve, that is, the smaller the circle, the less is the stability.***

442. In order that a vehicle may have the same stability in revolving on a circular road, which it possesses when moving on a straight and level one, it is obvious that such a slope must be given to the road surface, as that the intersection of the resultant pressure $O\,R$ may be midway between the points of contact A and B of the wheels with the road, as at x, (fig. 214.) The slope necessary to give this stability is thus found.

In the triangle $O\,Q\,R$, the line $O\,Q$ represents the weight of the vehicle, $Q\,R$ the centrifugal force, and $O\,R$ the action at right angles to the surface of the road. Using F, to denote the centrifugal force, and W the weight of the carriage, we have $F : W = Q\,R : O\,Q =$ tangent $Q\ O\,R : 1$. Hence tangent $Q\ O\ R = \frac{F}{W}$. But by similar triangles the angle $Q\,M\,B$ is equal to $Q\ O\ R$. Denoting these by φ, we have tangent $\varphi = \frac{F}{W}$. Substituting for F its value $W \cdot \frac{v^2}{g\,r}$, this gives tangent $\varphi = \frac{v^2}{gr}$. That is, in order that the stability of the vehicle, when moving around the curve, may be the same as on a horizontal plane, the tangent of the inclination of the road must be equal to the square of the velocity of the carriage, divided by 32 times the radius of the curve.

443. We may also find the slope by computing F the centrifugal force, and then, as in (440), constructing a triangle (fig. 215), whose perpendicular sides, OQ QR, have the direction and ratio of W and F, the third side $O\,R$ gives

the direction of the resultant pressure, and a line $M\,N$ at right angles to this will denote the slope of the road surface. The angle of inclination a, as already stated, is equal to the angle at O. Upon this principle is computed the height necessary to be given to the outer above the inner rail in the curves of railroads.

444. OF THE CENTRIFUGAL FORCE DUE TO THE ROTATION OF THE EARTH.

In virtue of this rotation, bodies on the surface of the earth are carried round the axis in parallel circles (fig. 216) once in 24 hours, with a motion which is perfectly uniform. The only points which do not thus revolve are N and S the ends of the axis, or the north and south poles. The velocity is evidently greatest at the equator $Q\ m\ q$, and less as we recede from this circle towards N or S. To compute the ratio of the centrifugal force at the equator to the force of gravity, we recur to the formula $F = W . \frac{v^2}{g\,r}$. Assuming the circumference of the globe at the equator to be 25,000 miles, the velocity of a body at the equator is found thus. As one mile = 5280 feet, we have 25,000 miles = 25000 × 2580 feet. This space is described in 24 hours = 24 × 60 × 60 seconds. Therefore the velocity $v = \frac{25000 \times 5280}{24 \times 60 \times 6} = 1530$ ft. per second. Again the radius of the earth $= r = 21{,}120{,}000$ ft. Substituting for v and r in the expression $F = \frac{v^2}{g\,r}$, we have $F = W \frac{(1530)^2\ (+)}{32 \times 21{,}120{,}000} = W \frac{1}{289}$ or $\frac{1}{289}\ W$.

That is, *the centrifugal force acting on a body at the equator, in virtue of the earth's rotation, is equal to* $\frac{1}{288}$ *of the weight of the body.*

445. As this centrifugal force at the equator acts in direct opposition to gravity, it diminishes the weight or downward pressure of a body to the amount of $\frac{1}{288}$ of the whole weight. In the same manner may be computed the centrifugal force at the equator of the sun, or of any of the planets, the rotary velocity and radius being known.

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446. Supposing the velocity at the equator to be increased, the consequent increase of centrifugal force would cause a still greater diminution of gravity, and we may compute on the principles above laid down, the precise rate of rotation at which the centrifugal force would balance the weight, and at which therefore a body would cease to exert any pressure upon the surface. The calculation is as follows:

Let v = the required velocity. In the formula $F = W\frac{v^2}{g\,r}$ put $F = W$, and we have $1 = \frac{v^2}{g\,r}$. Whence $v^2 = g\,r$ and $v = \sqrt{g\,r}$. But r = radius of the earth = 21,120,000 and $\sqrt{g\,r} = \sqrt{32 \times 21{,}120{,}000} = 26{,}000$. That is, *a velocity of* 26,000 *feet per second at the equator would generate a centrifugal force equal to gravity, and thus entirely prevent the downward pressure of bodies.*

If we suppose a ball to be impelled with this velocity horizontally near the earth's surface, and the atmosphere to oppose no resistance to its motion, it would revolve permanently in a circle around the earth, neither approaching the surface nor receding from it. For the centrifugal force in this case is equal to the weight, and therefore incessantly counter-balances the tendency to fall.

447. COMPARISON OF THE CENTRIFUGAL FORCE IN DIFFERENT LATITUDES. — Let F (fig. 217) denote the centrifugal force of a body at Q, in the equator, and F_1 that of an equal mass at P. As their circles of revolution $Q\,T\,q\,t$ and $P\,V\,p\,v$ are described in equal times, their centrifugal forces are proportional to the radii of the circles (431), that is, $F : F_1 = C\,Q : D\,P = C\,P : C\,R$. But $C\,P : C\,R = 1$: cosine L_1, the latitude. Whence $F_1 = F$. cosine L_1. That is, the centrifugal force of a body any where north or south of the equator is equal to the product of the centrifugal force at the equator into the cosine of the latitude. Assuming now some other latitude, L_2, and denoting by F_2 the centrifugal force, we have $F_2 = F$. cosine L^2; therefore, $F_1 : F_2$ = cosine L_1 : cosine L. That is, *the centrifugal forces at different latitudes are proportional to the cosines of the latitudes.*

448. OF THE VERTICAL ELEMENT OF CENTRIFUGAL FORCE. — While at the equator the whole centrifugal force acts vertically and in opposition to gravity, it is not so with this force in places north or south of the equator. Thus, while at Q, (fig. 218) the centrifugal force is in the direction, $Q\ A$, opposite to $Q\ C$, the direction of gravity; at P this force has the direction $P\ a$, while that of gravity is $P\ C$. Extending $P\ C$ beyond the circle and drawing the tangent at P, the centrifugal force F_1, represented by $P\ a$, may be resolved into $P\ b$ and $P\ c$. Of these, $P\ b$ is directly opposed to gravity at P, and $P\ c$, which is at right angles to gravity, neither helps nor opposes it. $P\ b$, the vertical element of the centrifugal force, is what measures its efficiency, in diminishing the weight of bodies placed at P. By similar triangles, $P\ a : P\ b = P\ C : P\ R = 1$: cosine L_1. Therefore, F_1 : vertical element $= 1$: cosine L_1. Therefore, the vertical element $= F_1$ cosine L_1. But $F_1 = F$. cosine L_1. Therefore vertical element $= F$. cosine^2 L_1. Whence we infer that *the vertical or efficient element of the centrifugal force varies as the square of the cosine of the latitude.*

449. EFFECT OF THE CENTRIFUGAL FORCE ON THE FORM OF THE EARTH. — The tangential or horizontal element of the centrifugal force, viz., $P\ C$, in (fig. 218), tends to move the particle or body P towards the equator, and in the other hemisphere, as at P_1, this element of the centrifugal force presses also towards the equator. If, therefore, the earth were liquid throughout, or every where covered by a deep sea, this pressure towards the equator, due to the tangential element of the centrifugal force, would withdraw the water from the polar regions, and heap it up at or near the equator, thus changing the figure of the earth from that of a sphere, to a spheroid, flattened at the poles, and protuberant at the equator, as represented in (fig. 219). But this transfer of material and change of form would not go on without limit. When the form of the spheroid became such that the resultant $P\ x$, produced by the combined action of gravity $P\ C$, and the centrifugal force $P\ a$, was perpendicular to the surface at P_1, each particle of water would press perpendicularly against the supporting surface, and would, there-

fore, be sustained at rest. The form, thus assumed, is called *the spheroid of equilibrium.*

450. The form of the spheroid of equilibrium, that is, the proportion between its diameters *N S* and *Q S*, will evidently depend upon the velocity of rotation. The actual dimensions of the earth, as deduced from measurements, are as follows, omitting decimals: —

Greater, or equatorial diameter,	7925	miles.
Lesser, or polar diameter,	7899	"
Difference, or polar compression,	26	"

This is the exact amount of compression which, by calculation, ought to result from the rotary velocity of the earth, supposing its parts to be free to yield to the combined action of the centrifugal force and gravity. But, from geology, we learn that at a former period the entire globe was in a state of fusion, and that even now the solid part of the earth is but a thin shell, inclosing a molten mass. We may, therefore, confidently infer that *the spheroidal form of the earth is the result of its rotation.*

451. Illustrations of Centrifugal Force. — The change of form produced by the rotation of a sphere of liquid or flexible material, is well exemplified by an appendage of the Whirling Table represented in (fig. 220). This consists of a number of thin slips of brass connected together at the ends and bent into a circular form, so that by their union they have the form of the intersecting meridians on a globe. An axis of metal passes through the points where they unite, and by fastening this to one of the spindles of the whirling table, the flexible sphere is made to revolve. On making this experiment, the sphere is seen to become more and more flattened at the poles, and protuberant at the equator as the velocity of rotation is increased.

452. A far more perfect means of studying experimentally the effect of centrifugal force on the form of a revolving mass, has recently been devised by Plateau.

For this purpose two liquids are used, one of which is an oil, and the other a mixture of water and alcohol, carefully prepared, so as to have precisely the same density as the oil. When the oil is poured into a large mass of the mixed liquid contained in a wide glass vessel, (fig. 221), it has no tendency either to sink to the bottom or to rise to the surface. Its gravity is precisely balanced on all sides, and in virtue of the mutual attraction of its parts it gathers itself together in the form of a sphere. By proper manipulation this sphere of oil may be thus collected in the centre of the vessel. A vertical rod connected above with a mechanism for rotation descends through the liquid, and carries about midway in the depth a thin circular disc of metal. When the adjustment is complete, this disc divides the suspended globe of oil into two equal parts, an upper and a lower hemisphere.

When the disc is made to revolve, it quickly imparts rotation to the oil, which immediately assumes a spheroidal shape. As the velocity is increased, the flattening augments at the poles, and the liquid accumulates more and more towards the equator. At length, when a particular velocity has been reached, a part of the matter at the equator is thrown off from the mass in the form of a ring, which continues to revolve around an inner spheroid composed of the remainder of the original sphere.

453. The water which moistens the surface of a grindstone adheres to it with considerable force, but when the stone is turned very rapidly, the centrifugal force, which increases as the square of the velocity (428) becomes greater than the force of adhesion, and the water is seen to fly off in tangent lines from the different points of the circumference.

454. Millstones are sometimes seen to burst into fragments in consequence of too great velocity of rotation. In this case the centrifugal force which the rotation generates in the parts of the stone near the rim, is more than equal to the cohesion of the cement used in binding the blocks together, and is even sufficient to overcome the tenacity of the strong iron hoops which encircle the millstone.

455. One of the most interesting of the effects of centrifugal force is seen in the formation of *eddies* or *whirlpools* in liquids. Whenever a mass of water or other liquid is made to revolve, its surface ceases to be horizontal and assumes a concave form, which grows deeper as the velocity of rotation is increased. This is well seen when with an immersed stick we give rotation to water in a vessel; and the effect is grandly displayed in the great whirlpools of seas and rivers, some of which, as the Maelstrom, are revolving masses of water of vast extent.

The cause of the concave form assumed by the surface may be thus explained. Let HR (fig. 222) be the horizontal level of the water, a part of which at and near S is made to revolve rapidly around a vertical line. The revolving particles, P Q, &c., in consequence of this rotation, will exert a centrifugal effort or pressure at each point of their respective orbits. This being at first unbalanced, they will continue to recede from the axis and to widen their orbits until the counter pressure of the external water, which tends to force them back, is at last in equilibrium with their centrifugal force.

In this condition of moving equilibrium, each particle, as P Q, &c., on the surface of the cavity which has been formed, exerts a pressure, PR, due to the combined action of its weight, PW, and its centrifugal force PQ. So long as the velocity of rotation is kept up without change, and each particle continues to revolve in the same horizontal orbit, the pressure PR can have no tendency to shift the particle either upwards or downwards on the surface of the curve. Hence PR is perpendicular to the curve at P; and thus in general when the surface of the rotating liquid has assumed its permanent form, or what may be called its figure of equilibrium, the pressure exerted by each particle is perpendicular to the element of the surface which it touches. This is but an extension of the law already proved in the case of the spheroid of equilibrium.

If the velocity of rotation be increased, the concavity of the liquid must become deeper. For PQ, the centrifugal force, is increased, while PW, the weight of a particle, remains the same. Hence the new resultant $P_1 R_1$ must approach nearer to a horizontal direction. But the element of the curve in the permanent state of the surface is perpendicular to $P_1 R_1$. Hence the curve must adjust itself to a steeper inclination, as at P_1 (fig. 223).

CHAPTER XXIII.

MOTION OF PROJECTILES.

456. We have already seen that a body projected in a horizontal or oblique direction at the earth's surface, will be continually deflected from its straight path by gravity, and will thus describe a curve intersecting the earth's surface at a greater or less distance from the point of projection, according to the velocity and direction of the original motion.

The force of gravity acting upon the projectile at different points of its path, is continually changing both in direction and intensity. Thus (fig. 224), its directions at A V and B are AS, VS and BS, and its intensity at V is less than at A or B. As, however, the greatest horizontal distance, AB, to which a projectile can be thrown, is only some two or three minutes of a degree, we may without sensible error assume SA, SV, SB, to be parallel. Again, the greatest height, CV, is in no case more than one mile, and thus by (341) the diminution of gravity can never amount to $\frac{1}{2000}$th part of the whole force. We may, therefore, regard gravity as constant at all points of the path. The resistance which the atmosphere opposes to the motion of a projectile greatly modifies its path; but for the present we will leave this out of consideration, and regard the projectile as moving in a vacuum.

In treating of the Theory of Projectiles, we will, therefore, make the following assumptions:

1st. That the force of gravity acts on the projectile in parallel directions at all points of its path.

2d. That it has the same intensity at all points.

3d. That the projectile moves in a vacuum.

457. To show in what way the path of the projectile may be graphically traced, we begin with the case of motion in a horizontal direction.

Let AT (fig. 225) be an indefinite horizontal line representing the motion of projection. This motion alone would carry the body over equal successive spaces in equal times. Let $A\ 1''$ represent the known velocity of projection, and take spaces $1''\ 2''$, &c., equal to this along the line $A\ T$. At the end of 1, 2, 3, &c. seconds, the body would be at the points $1''\ 2''\ 3''$, &c. But by the uniform action of gravity, the body in 1 second will have fallen 16 feet, in 2 seconds 4×16 feet, in three seconds 9×16 feet, in t seconds $t^2 \times 16$ or $\frac{1}{2} g t^2$ feet. Hence, drawing the verticals from $1''\ 2''$, &c., to represent these distances, the points $B\ C\ D$, &c. will mark the successive positions of the projectile in its curved path at the end of 1 2 3, &c. seconds.

When the line of projection is oblique, as in (fig. 226) we proceed in the same way, laying off equal distances, $A\ 1''$ representing the uniform velocity of projection, and verticals representing on the same scale the deflecting action of gravity in 1 2 3 t seconds. The points $B\,C\,D$, &c. will mark the position of the projectile in its curve at the end of 1 2 3, &c. seconds.

By dividing the time into intervals much smaller than a second, and laying off as above the spaces corresponding to the projectile motion and the vertical deflections, we may determine any number of points in the curve described by the projectile (figs. 225 and 226.)

From the above, we deduce the following rule for finding the position of the projectile at any given moment, supposing the direction and velocity of its projection to be known. Find its place on the line of projection at the given instant, compute the vertical descent due to the time elapsed, and lay off this computed distance vertically beneath the place first found. The true position of the body will be at the lower end of this vertical line.

458. TO DETERMINE THE NATURE OF THE CURVE DESCRIBED BY A PROJECTILE. — Simple inspection of the above figures is enough to show that this path is not a circle, but a curve, having most rapid inflection at V, and becoming more nearly a straight line as it is prolonged on either side. Let $A\ B$ (fig. 227) be the path of a projectile, $A\ T$ the line of projection, and v the velocity impressed in that direction. Let t be the time in which $A\ M$, any assumed distance on $A\ T$, would be described with the velocity of projection v. Then $A\ M = v\ t$ and $A\ M^2 = v^2\ t^2$. Drawing the vertical $M\ P$,

this will represent the space through which gravity has drawn the body downwards in that time. Hence $MP = \frac{1}{2} g t^2$. From the former expression we have $t^2 = \frac{AM^2}{v^2}$. This substituted for t^2 in the latter, gives $MP = \frac{g}{2v^2} AM^2$. So for any other point P, we have $M_1 P_1 = \frac{g}{2v^2} AM_1^2$. Hence $MP : M_1 P_1 = AM^2 : AM_1^2$. That is, the curve is so inflected that the tangent increases as the square of the vertical distance from it to the curve.

This relation characterizes the curve called the Parabola. Hence *the theoretical path of a projectile is a parabola.*

459. The time occupied by the projectile in describing the curve from A to B on the same horizontal plane, (fig. 228) is called the *time of flight.* The distance AB from the point of projection and the point where the body strikes the same horizontal plane, is called the *horizontal range.* CV is called the *vertical range* or flight.

The velocity and angle of projection being given, we can readily find each of the three quantities above named, as shown in the following problems.

460. Problem first. — *To find the time of flight.*

In (fig. 228) let AD be the line of projection, making with AB the given angle of projection φ. Assume AR equal to v, the velocity of projection. Draw RP and RQ at right angles to AB and to the vertical AK. Then AQ and AP will be the vertical and horizontal elements of the velocity v. These may easily be determined from AR or v and the angle φ.

In ascending from A to V, that is, in half the time of flight, gravity destroys the vertical velocity. But the time in which any vertical velocity is destroyed by gravity, is equal to this velocity divided by g, the velocity which is destroyed in 1″.

Therefore half the time of flight $= \frac{\text{vertical velocity}}{g}$,

and whole time of flight $= \frac{2 \text{ vertical velocity}}{g}$.

That is, *the time of flight is found by dividing twice the* *vertical element of the velocity of projection by the acceleration g.*

Example. Let $v = AR = 1600$ feet per second, and angle $\varphi = 30°$, then vertical velocity $= AQ$, will be 800 feet per second, and horizontal velocity $= AP$ will be 1385 feet per second. The time of flight will be

$$\frac{2 \times 800}{32} = \frac{1600}{32} = 50 \text{ seconds.}$$

461. Problem second.—*To find the vertical range CV.*

This is the height through which the body impelled by the vertical velocity is capable of rising, and is equal to the distance through which the body would have to fall in order to acquire the vertical velocity. Referring to the formula $v = \sqrt{2gs}$, we have vertical velocity $= \sqrt{2g \times VC}$ and $VC = \frac{\text{vertical velocity}^2}{2g}$. That is, the *vertical range is found by dividing the square of the vertical velocity by $2g$.*

Example.—In the preceding case the vertical velocity $= 800$. Hence $VC = \frac{800^2}{64} = \frac{640000}{64} = 10{,}000$ feet or $1\frac{9}{10}$ miles nearly.

462. Problem third.—*To find the horizontal range AB.*

The horizontal element of the velocity of the projectile being at right angles to gravity, is neither aided nor opposed by that force, and therefore remains constant throughout the flight. Thus while rising from A to V and then descending from V to B, the projectile is at the same time moving forward with a uniform horizontal velocity, viz., the horizontal element of the projectile motion. The space AB may therefore be regarded as described in the time of flight and with the uniform horizontal velocity.

Hence, the horizontal range is found by multiplying the time of flight by the horizontal velocity.

Example.—In the preceding case the horizontal velocity $= 1385$ feet per second, and the time of flight as computed in (460) is $= 50$ seconds. Hence $AB =$ horizontal range $= 50 \times 1385 = 69{,}250$ feet $= 11\frac{2}{10}$ miles.

463. Substituting for the time of flight the value obtained

(in 460), viz., $\frac{2.\text{vertical velocity}}{g}$, we have horizontal range $= \frac{2}{g} \times$ vertical velocity $\times$ horizontal velocity.

Let $h =$ the height through which the projectile would have to fall in order to acquire the velocity v. Then $v = \sqrt{2gh}$, and $v^2 = 2gh$. Therefore $g = \frac{v^2}{2h}$. Substituting this value of g in the above expression for the Hor. Range, we have Hor. Range $= \frac{4h}{v^2} \times$ vertical velocity $\times$ horizontal velocity.

464. This suggests the following simple geometrical construction, for determining the horizontal range when the velocity and angle of projection are known.

Construction. — Having computed the value of h in feet from the equator $v^2 = 2gh$ in which v and g are known, assume the horizontal and vertical lines AH and AK, (fig. 229) and on the latter lay off AL to represent $4h$. Describe a semicircle on this line, and draw AT in the direction of the initial projectile motion. From the point M, where this cuts the circle, drop the vertical MB. The distance AB will be the horizontal range.

Proof. — Draw LM. Then v : vertical velocity $= (AM : BM) = 4h : AM$. and $AM = \frac{4h.\text{vertical velocity}}{v}$ Also v : horizontal velocity $= AM : AB$. Whence $AB = \frac{AM \times \text{horizontal vel.}}{v} = \frac{4h}{v^2} \times$ vertical velocity $\times$ horizontal velocity. Therefore $AB =$ horizontal range.

Thus, a projectile impelled with a given velocity will have the same horizontal range, whether its line of projection makes a given angle with the horizon, or the same angle with the vertical.

465. Problem fourth. — *To find the direction producing the greatest horizontal range.*

By increasing the angle BAM or φ to BAM_1 (fig. 230) the horizontal range is increased to AB_1 and the points M m are made to approach. When $\varphi = SAB$ these points coalesce at S; the vertical line becomes a tangent to the curve at S, and the horizontal range $= A\,x$ is the greatest possible. The two tangents $A\,x$ and $S\,x$ are equal (by geometry), therefore the angle $S\,A\,x$ is equal to the angle $x\,S\,A$. But the angle at x is 90°, therefore the sum of $S\,A\,x$ and $x\,S\,A$ is a right angle, and therefore the angle $S\,A\,x = \varphi = 45°$. That is, *with a given velocity of projection the horizontal range is greatest when the angle of projection is* 45°.

Since $A\,x = x\,S = \frac{1}{2}\,L\,A = 2\,h$, it follows that the greatest horizontal range is equal to twice the vertical height due to the velocity of projection.

Example. — Let v, the velocity of projection, be 1600 feet per second, we have $v^2 = 2\,g\,h$ and $h = \frac{v^2}{2g} = \frac{1600^2}{64} = \frac{256000}{64} = 40{,}000$ feet and greatest horizontal range $= 2\,h =$ 80,000 feet $=$ 15 miles.

466. To FIND THE INITIAL VELOCITY OF SHOT AND OTHER PROJECTILES.

Various methods have been contrived for this purpose, of which the most remarkable are the following.

First. —*By the Revolving Drum.* $M\,N$ (fig. 231) is a cylindrical drum made of thin paper attached to a light frame. This is kept in uniform and very rapid rotation around its axis during the experiment. The pistol or rifle ball impelled in the direction $R\,a$ so as to pass through the centre S, will describe a diameter of the drum, making a hole in the paper at its entrance and at its exit. Supposing the drum at rest, the second perforation would obviously be at b, diametrically opposite to a. But the motion of the drum during the transit of the ball carries the point b to some other position, as b_1, and brings x to the position b. Thus a and x mark the points of entrance and exit of the shot. Subtracting $a\,x$ from 180°, we have $x\,b$ the arc of rotation described in the time occupied by the shot in crossing the drum. Knowing the rate of rotation of the drum, it is easy to compute the time of describing $x\,b$, and this is the time taken by the shot to describe the diameter $a\,b$ of the drum. Let this be represented by t, then v the velocity of the shot will be $= \frac{a\,b}{t}$.

Example. — Let the diameter be 2 feet, the time of revolution $\frac{1}{100}$ of a second, and let $x\,b = 60° = \frac{1}{6}$ of a revolution, then $t = \frac{1}{600}$th of 1″, and $v = 2$ divided by $\frac{1}{600} = 1200$ feet per second.

467. Second. — *By the Recoil.* In this method, the gun is suspended horizontally by a rigid bar from a pivot S, (fig. 232) and an index is attached to its under surface to mark, on the graduated scale beneath, the exact distance of recoil. The expansive force of the exploded powder acts equally, but in opposite directions, upon the gun and the ball, making the momentum with which the gun recoils equal to the momentum with which the ball advances. Let $G =$ the weight of the gun, and B the weight of the ball, also let $v =$ the velocity with which the gun recoils, and $x =$ the velocity imparted to the ball. Then we have $B\,x = G\,v$ and $x = \frac{G\,v}{B}$. That is, the velocity of the ball is equal to the product of the weight of the gun into the velocity of the gun divided by the weight of the ball.

To be able to compute x, we must find the value of v, the velocity of the gun, which is done thus. From the observed arc of recoil $V\,A$, we find the vertical ascent $V\,C$ of the centre of gravity of the gun. It has been shown under a former head, that the velocity necessary to carry a body from V up to A, is equal to that which would be acquired in descending from A to V, and therefore equal to the velocity acquired by falling from C to V. Hence $V = \sqrt{2\,g \times C\,V}$. Having thus determined v, and knowing G and B the weights of the gun and ball, we readily compute x, the velocity of the ball. In the above, we suppose the weight of the suspending rod to be insignificant compared with that of the gun. This method has been used by Count Rumford and others, for finding the velocity of musket and rifle bullets. It is not well suited for experiments with cannon.

468. Third. — *By the Balistic pendulum.* This consists of a heavy block of wood, or a block of metal suspended by a strong rod on a horizontal knife-edge (fig. 233). When wood is used, the impinging ball buries itself in the mass and vibrates with it. When the block is of metal, a cavity is left in the face of it, which is filled with clay or other

inelastic substance, into which the ball may sink. Projecting from the block below is an index or pointer, which marks the extent of the vibrations of the pendulum, by leaving its trace on a thin layer of wax with which the concave surface of the graduated arc is coated for the experiment. The ball is usually discharged so as to strike the pendulum horizontally in a line with its centre of oscillation. The arc of vibration being measured, we compute from this the weight and figure of the pendulum, as well as the weight of the ball, the velocity of the ball applying the formula provided for the case, but which is too complex to be given here. If the whole matter of the pendulum were collected at the point struck by the ball, we might find v the velocity of the pendulum from the height of its arc of vibration, as in (468.) Call G the mass of the pendulum, B that of the ball, and x the velocity of thelatter. Then we have $B\,x$ equal to the momentum of the impinging ball, and $(G + B)\,v$ equal to the momentum of the united mass B and G after the impact. But this is equal to the former, that is, $B\,x = (G + B)$. Therefore $x = \frac{G + B}{B}$.

469. OF THE EFFECT OF THE RESISTANCE OF THE AIR UPON THE PATH OF A PROJECTILE.

As the body, in its motion through the air, is incessantly pushing the particles of this medium out of its way, and as by the equality of action and reaction, the force exerted by the air against the body is equal to that which is expended in displacing the air, it follows that the momentum lost by the projectile in a given time is equal to that which in the same time has been imparted to the air. This reaction of the particles of the air, or other medium traversed by a moving body, is what is called the *Resistance of the medium.*

Obviously this resistance will be greater, in proportion as the medium is more dense, because the moving body will then be compelled, in a given time, to displace a greater quantity of matter, and will therefore sustain a greater reaction. With the same velocity, a body moving in a medium of double density will sustain a double resistance, and in general, other conditions being the same, *the resistance is in proportion to the density of the medium.*

An increase of the surface exposed to the resisting force must in like manner increase the resistance. Thus a sphere of double surface, moving with a given velocity in a given medium, will encounter twice the resistance sustained by one of half its surface. That is, other conditions being the same, *the resistance is proportional to the surface exposed to the resisting force.*

470. OF THE LAW OF RESISTANCE AS DEPENDENT ON THE VELOCITY OF THE PROJECTILE. — In the investigation of this subject two methods are followed. First, that of theory, founded on certain assumptions in regard to the nature of the medium; and Second, that of experiment, in which the actual resistance due to different velocities is ascertained. The results of the two methods are in close agreement only when the velocity of the projectile is small.

471. *First, or Theoretical Method.* — In this we assume that the particles of air lying in the path of the moving body cease after their displacement to exert any further action, and that the air sustains neither compression nor rarefaction from the transit of the projectile.

Let A, B, C, (fig. 234) be equal spheres, moving through the air with the velocities 1, 2 and 3 feet per second respectively. In one second B displaces twice as many particles, and C displaces three times as many particles as A. Moreover, the velocity which B gives to each particle is twice that which A imparts. Therefore the whole momentum given to the air by the motion of B for one second, is four times that given in the same time by the motion of A. So the momentum given to the air by C in one second, is nine times that given by A. But the momentum with which the air is displaced (469) is equal to the momentum lost by the projectile; that is to say, the resistance. Hence the resistances sustained by the equal balls A, B, C, moving with the velocities 1, 2, 3, are in the proportion of 1, 4, 9.

That is, in general, *the resistance sustained by a ball or other body moving through the air, is proportional to the square of its velocity.*

Example. — It has been found by experiment that a ball 6⅜ inches in diameter, moving with a velocity of 16 feet per second, sustains a resistance of ⅔ of an ounce. What would the resistance amount to with a velocity of 1600 feet per second, according to the law above stated?

Here $16^2 : 1600^2 = \frac{2}{3} : R$ the resistance in ounces. Hence $R = \frac{2}{3} \times \frac{1600^2}{16_2} = \frac{2}{3} \times 100^2 = 6666$ oz. $= \frac{6666}{16}$ pounds $= 416$ pounds. Enormous as is this estimate, we shall presently see that the actual resistance is more than twice as great.

472. Effect of the condensation in front of the ball and the rarefaction behind it. — When the motion of the projectile is slow, the particles of air in its path have time to escape and give place to others, and there is no perceptible crowding or condensation of the air in front of the body. So the air adjoining the body, laterally and behind, has time to fall in and fill the space which the body successively leaves as it moves on, and thus the air behind as well as in front retains very near its natural density. But when the velocity is considerable, this ceases to be the case. The air in front of the body becomes crowded or condensed, and that in the rear has not time by the influx from around to resume its natural density, and is therefore rarefied. Both these changes conspire to increase the resistance to the motion of the projectile.

If P denote the pressure of the condensed air in front of the ball, and p that of the rarified air behind, then since the former opposes and the later helps the motion, the whole resistance due to the unequal density of the air is measured by $P - p$. It will hereafter be proved that the greatest velocity with which the air can flow into the spaces successively left by the ball, is about 1300 feet per second. Hence, if the velocity of the ball exceed this amount, these spaces will be left by the ball faster than the air can flow in to fill them, and there will be a vacuum immediately behind the ball. As the ordinary rate of motion of cannon and musket balls exceeds this amount, it follows that in such cases there is a vacuum in the rear of the projectile. Thus the pressure p disappears, and the resistance due to unequal density is proportional simply to P, the pressure in front of the ball.

What may be the amount of this pressure P with a given velocity, we have no means of determining. But we know that at high velocities it must be very great. Even supposing no condensation in front of the ball if there be a vacuum behind it, the resistance due to *P* will be the ordinary pressure of the atmosphere or 15 lbs. per square inch. This in the case of the ball, whose diameter is 6⅜th inches, will give an aggregate pressure *P* of about 560 pounds. Adding this to the resistance caused by the displacement of the air, as computed by the rule of the square of the velocity (471), we have a total of 976 pounds. But the condensation in front of the ball would increase *P* greatly beyond the amount above assumed, so that 976 pounds is much below the actual resistance of a ball of 6⅜ inches in diameter, moving through the air with a velocity of 1600 feet per second.

473. *Second. Experimental method of determining the resistance.*

Various mechanical arrangements have been used for finding the resistance of flat surfaces, spheres and other forms, at different but moderate velocities. For high velocities, as in the case of cannon-balls, the Balistic pendulum affords the best means of ascertaining the resistance. With this view, equal balls propelled by equal charges of powder are shot successively against the bob of the pendulum from stations more and more remote, say 10, 60, 110 feet (fig. 235), and their velocity at the instant of striking the pendulum is computed, as explained above, (468). They will, of course, have equal velocities at these stations on leaving the piece. But on reaching the pendulum the ball from *A* will have a greater velocity than that from *B*, and the latter a greater velocity than the ball from *C*. This progressive reduction of velocity is obviously due to the resistance of the air between *B* and *A*, and between *C* and *B*.

Let *v* denote the loss of velocity sustained by the ball in passing through the distance *B A*, and let *t* be the time of transit from [illegible] *A*. To find this time, we regard the space *B A* a[illegible]ribed with a uniform velocity, which is a mean between the velocity of the ball when first discharged at *B*, and the diminished velocity with which it passes through *A*. As the pendulum is placed very near the point *A*, the velocity with which the ball impinges upon it may be assumed to be that with which it escapes from the cannon;

and this is equal to the initial velocity at B when the mouth of the piece is removed to that position. The diminished velocity at A is in like manner assumed to be equal to the velocity with which the ball discharged at B strikes the pendulum, placed at the same station as before. Knowing thus the initial velocity at B, and the velocity after the action of atmospheric resistance through BA, we take half their sum for the average velocity of the ball along BA. Dividing BA by this quantity, we have $\frac{BA}{\text{velocity in } BA} = t$, the time of describing BA.

In the time t thus computed, the resistance of the air destroys the velocity v. But we know that in the time t the force of gravity would destroy the velocity gt. We may, therefore, compare these two forces by the ratio of the effects thus determined, and find the value of the resistance in terms of gravity or the weight of the ball. Let R denote the resistance and W the weight. Then we have $R : W = v : gt$.

Therefore $R = W . \frac{v}{gt}$.

Example. — In an experiment of Robins's, a rifle-ball weighing one ounce, in passing through $AB = 50$ feet with an average velocity of 1600 feet per second, lost a velocity of 123 feet per second. Here $W = 1$, $v = 123$ and $t = \frac{50}{1600} = \frac{1}{32}$. Hence $gt = 32 \times \frac{1}{32} = 1$. Therefore $R = W . \frac{v}{gt} = 123$ ounces. That is, *the resistance of the air in the first period of the motion was* 123 *times the weight of the ball.*

474. The following table of the resistances sustained by a ball of 2.78 inches diameter, weighing 3 lbs. and moving with different velocities, may serve as an example of the results obtained by this method.

Velocity. —	900 feet.	1000.	1100.	1200.	1300.	1400.	[illegible]	[illegible]	1700.	1800.
Resistance. —	35 pounds.	47.	60.	74.	91.	112.	[illegible]	[illegible]	165.	176.

475. The following case illustrates the extent to which the actual motion of projectiles differs from their motion, computed on the supposition that they meet with no resistance.

A 24 pound ball impelled with the velocity of 1640 fee per second would, according to this hypothesis, have for its maximum horizontal range $2h = \frac{v^2}{g} = 84050$ feet, and this would take place at an angle of 45°. But from experiments and computations which take account of the resistance of the air, it has been proved by Hutton that the greatest range in this case would actually be 7829 feet, and the angle giving this range 34° 15′. We thus have the following comparison:

(1). Omitting resistance of air, *H*. *R* = 84050 feet. Angle of maximum = 45°.

(2). Including resistance of air *H*. *R* = 7829 feet. Angle of maximum = 34° 15′.

476. Effect of Rotation in deflecting the path of Projectiles. — Confining our notice in this place to the motion of balls, it is obvious that when the projectile moves forward *without revolving*, (fig. 236) the sides *A* and *B* and the upper and lower parts at and opposite to *C*, pass by the contiguous air with precisely equal velocity, and must, therefore, sustain the same amount of resistance from successive contact with it. So, likewise, when the ball rotates around an axis *P Q*, coinciding with the direction of the forward motion. For in this case (fig. 237) we see that the rotary motions of the parts *A*, *B*, &c., are at right angles to their progressive motion, and, therefore, neither help nor oppose this motion. Hence *A* and *B*, &c., strike the air with equal velocity as they move on.

But ~~if~~ the ball *rotate around an axis at right angles to the line of progress*, the parts *A* and *B* will impinge with unequal velocities on the contiguous air. In (fig. 238) the projectile and rotary motions of *A* are in the same direction, and those of *B* are in opposite directions. Hence the velocity with which *A* strikes the air is the sum of the two velocities, and that of *B* their difference. But as the greater velocity generates a greater resistance, the action of the air at *A* exceeds its action in the opposite direction at *B*, and the ball is therefore deflected from its straight path *H R* into the line *H S*. When the rotation is reversed, as in (fig. 239) the ball is made to deviate towards the other side. In general, the ball is deflected towards the side at which the rotary motion is backward. The greater the velocity of rotation, and the rougher the surface of the ball, the greater will evidently be the deflection due to this cause.

477. If *the projectile be irregular* in form, as a ball with one or more prominences (fig. 240) at one side, it will deviate towards the opposite side, even should it have no rotation. But if it be made to rotate around an axis coinciding with the line of progress, its deviations being always *from* the side where the prominence is, will carry it *successively to the right and left, and upwards and downwards*, and thus it will describe a spiral path through the air, equidistant at all points from the axis or line of projection, and, therefore, maintaining the same general direction.

From what has been stated, it follows that, to prevent an irregular wandering of the ball, it must, in the act of projection, be made to rotate round an axis in the direction of the gun. This is effected by a spiral groove in the bore of the piece, which compels the shot to screw its way out to the muzzle, and thus to pass into the air with a rotation which prevents any continued deviation from the vertical plane of its projection.

The same effect is produced by the oblique feathers of a shuttle-cock, (fig. 241) which give it rotation, and thus, by alternate small deflections to the right and left, prevent any marked departure from the general plane of projection.

MOLECULAR MECHANICS.

CHAPTER XXIV.

OF FRICTION.

478. If we suppose two bodies to be in physical contact through the medium of surfaces which are perfectly smooth, it is evident that the movement of the one body upon the other would not require any more effort than its movement when at a distance from the other. For in this case the only molecular action which could take place between them would be in a direction at right angles to the common surface of contact, and this would neither oppose nor aid the force applied. But in reality the surfaces of bodies, however smooth they may seem, are covered with asperities. The art of polishing is only a means of diminishing the size, while it multiplies the number of these ridges and furrows with which all surfaces are covered. When, therefore, one body presses on another, in virtue either of its weight or an external force, the prominences and hollows of the contiguous surfaces become more or less interlocked. In moving the one upon the other, pressures and collisions arise between the opposing asperities, and thus a resistance is produced parallel to the common surface of contact. *This resistance is called Friction.*

479. Friction operates not only when one surface is actually moved upon another, but when the applied force is insufficient to cause any visible motion. In this case the interlocking asperities are pressed more closely together in the direction of the applied force, exciting a reaction in the opposite direction. This kind of friction or reacting pressure augments with the force used, up to a certain limit when any further increase of the applied force is productive of motion. At this limit the resistance is just equal to the

moving force. The resistance in this case is called the *Friction of repose.* The resistance brought into action while one surface is moving on the other is called the *Friction of motion.*

480. A knowledge of the friction of repose is important in computing the forces which maintain a machine in equilibrium. If we suppose the power applied to a machine, say a wheel and axle, to be the greatest it will bear without being moved, this power P must be such as to balance the weight W, together with the friction of repose at the pivot C. Again, a knowledge of the friction of motion is necessary when we are to compute the motions and mechanical work of a machine from the motive power and other data. Thus, if P give motion to the wheel and axle, its power to raise a weight or do other work cannot be found without first knowing how much of it is exhausted by the friction of motion.

481. THE FRICTION OF REPOSE IS GREATER THAN THAT OF MOTION. — This is proved by experiment, and would naturally be inferred from the fact, that by continued rest the irregularities of the adjoining surfaces are enabled to become more closely interlocked than when one surface is moved upon the other. With very hard bodies, such as steel, the difference is much less marked than with the various kinds of wood.

It is doubtless owing to the same cause that in many instances a greater force is required to move one surface upon another after a prolonged than after a brief contact. This increase of the friction with the duration of the contact, does not cease in the case of some woods until after several hours; while in the hard and polished metals the maximum is attained, almost immediately.

482. A slight tremor given to the surfaces will suffice to release them in part from their connection, and to cause the friction of repose to pass into that of motion. Thus, a mass on an inclined plane, kept from descending by the friction of repose, will, by a light tap of the finger on the plane, be set in motion, and will then continue to descend.

Friction is of two kinds, according as the motion which excites it is that of a sliding or a rolling body. The former is called SLIDING FRICTION, and the latter ROLLING FRICTION.

483. *Sliding friction* operates not only between plane surfaces, as when the mass *W* is drawn along the horizontal table *A B* (fig. 242), but when a wheel revolves on a *fixed axle*, as in the case of a carriage wheel, and when an axle revolves *in gudgeons* at its ends, as in a mill-wheel. The resistance at *pivots* in watch-work and other mechanism, is also of this kind.

484. *Rolling friction* is best seen in the case of a wheel or cylinder (fig. 243), rolling on a plane *H R.* Each point of the circumference is successively brought to touch the plane; so that taking *b* and *c* for the points where *B* and *C* touch in the course of the rolling motion, we have the straight lines *A b* and *A c* equal to the arcs *A B* and *A C.* This is the kind of friction that acts at the rim of a carriage wheel when it touches the road, while sliding friction is exerted at the axle.

In general, rolling friction is much less powerful than sliding friction. This arises from the fact, that the parts of the moving surface are successively lifted from the parts of the surface on which the body rolls. We may illustrate this action by what occurs when a toothed wheel is moved along a rack, (fig. 244). The teeth of the wheel which are inserted between those of the rack are, by the forward motion, pressed against the posterior slopes of the latter, and as this motion continues, the teeth of the wheel instead of sliding over the rack are made to revolve backwards, and are thus obliquely lifted, while other teeth towards *S* are brought down upon an advanced part of the rack. There is thus a great deal *less direct collision between the asperities of two surfaces in the case of a rolling than that of a sliding motion.*

485. OF THE EFFECT OF UNGUENTS, SUCH AS OIL, TALLOW, TAR, &c. — It is well known that these substances, when interposed between the rubbing surfaces, have the effect of *diminishing the friction.* This result appears to be chiefly due to a filling up of the hollows and pores by the unctuous matter, forming as it were a new surface, much smoother than the original one. The same thing occurs when *black lead* or *soapstone* is used for diminishing the friction of machinery.

As both the tenacity of the unguent itself and its adhesion to the adjacent surfaces are resisting forces, they add to the effect of friction. But in most cases their effect is so small, compared with the friction, as to be practically insignificant. In delicate mechanism, however, such as watches and clocks, the thickening of the oil, either by oxidation or cold, augments the resistance to a serious extent, so as to retard or even stop the motion of the machine.

486. Three distinct conditions of surface are to be marked in sliding friction.

First. When *no unguent* has been applied to the surfaces.

Second. When the surfaces are simply *unctuous.*

Third. When *a layer of unguent* is interposed between them.

In the first condition the friction is greatest, in the second it is less, and in the third least of all.

OF THE LAWS OF SLIDING FRICTION.

The numerous and varied experiments of Vince, Coulomb, Morin and others, have established the following general facts or laws in regard to sliding friction.

487. Law I. — THE FRICTION BETWEEN TWO SURFACES IS PROPORTIONAL TO THE FORCE WITH WHICH THEY ARE PRESSED TOGETHER IN THE PERPENDICULAR DIRECTION. That is, with a double pressure the friction is double, with a triple pressure it is triple, and so on.

First method of Proof. — Let *A B*, (fig. 245), be a table with a level and smooth top, either of wood or metal, and *W* a body resting upon it by a flat surface, either like or unlike that of the table. Let a cord, fastened to the front of the body, be extended horizontally, so as to pass over the pulley *C*, and support the scale-pan *D*. If now weights be gradually added at *D* until the mass *W* is barely made to move, the whole weight, that is, the weight added in the scale plus that of the scale and cord will measure the friction of repose. Let *P* denote this weight or force. Next let the pressure of the sliding body upon the table be doubled, tripled, &c. successively, by weights placed upon it, and we shall find that the forces necessary to give it motion will be 2 *P*, 3 *P*, 4 *P* &c., or quantities approaching nearly to these.

If, while making the experiment, we give a slight tremor to the table, we will find that a much less force P suffices to cause the motion. Repeating the experiment in this way for double or triple pressures of W, it will be seen that the friction augments exactly in proportion to the pressure. The law thus proved is observed precisely and uniformly in cases like the last;- that is, where the *friction of motion* operates, but is liable to various and uncertain modifications in the case of the friction of repose. Fortunately, however, in practical cases of equilibrium, as well as of continued motion, it is the friction of motion with which we have to deal. For the slight tremor which is sufficient to bring this kind of friction into play, even in a structure whose parts are at rest, is almost certain to be one of the results of the balancing forces applied to it.

Second method of Proof.— Another, and perhaps better, mode of proof is the following: Let the cord, (fig. 24), by which the sliding body W is moved, be attached to a spring balance $M\ V$, and connected with some mechanism for drawing the body along the table. Adjust the motive force so as to cause a uniform motion of W, and note the tension in pounds and ounces indicated by the spring balance. This tension P is the exact measure of the friction; for the friction is constantly pulling in a direction opposite to that in which the tension draws the body, and as the motion is uniform, neither force can exceed the other. Let now the pressure of W be made double or triple, while the same uniform motion is maintained. It will be seen that the spring balance takes a new adjustment, marking a double or triple tension, and thus proving that the friction is increased just in proportion to the increase of the pressure.

Third method of Proof. — The same principle may be proved by placing the sliding body upon an inclined plane $A\,B$ (fig. 247), capable by a hinge at B of being adjusted at any required angle with the base $B\ C$, and then raising the plane gradually, so as to mark the inclination at which the body begins to move. If the weight of W be successively made double or triple of what it was at first, it will be found that the *same inclination* will just suffice to set it in motion.

In these experiments the force down the plane may be assumed as equal to the friction. But we know that Force down the plane : Pressure on the plane $= G H : H W = A C : B C$. Hence Friction : Pressure $= A C : B C$. But since with the same surface and varying weights, the inclination is found to be the same, we have $A C$ and $B C$ constant, and hence the proportion of the friction to the pressure must be constant. Thus, a double pressure perpendicularly against the plane is accompanied by a double friction, and a triple pressure by a triple friction, &c.

The truth of the first law of friction is evidently due to the fact, that the *greater the mutual pressure of the surfaces, the more closely do they become interlocked.*

488. Law II. THE FRICTION IS INDEPENDENT OF THE EXTENT OF THE RUBBING SURFACE.

First method of Proof. — This law may be demonstrated by using a sliding mass having one side much wider than the opposite, as shown in the plane and sections (fig. 248). If placed on the horizontal table (fig. 249) with the wide face *r s* beneath and then in the reverse position, it will be found to produce the same friction as long as the vertical pressure is kept the same. This result will present itself also when the experiment is made on the inclined plane. The body will begin to move down the plane at the same angle of adjustment whether resting on its wide or on its narrow surface.

Second method of Proof. — The second law may be deduced from the first in the following manner. Let W denote the weight of *m n r s* (fig. 248), and suppose the surfaces *r s* and *m n* to contain 10 square units and 2 square units respectively. The pressure upon a unit of *m n* is equal to $\frac{1}{2}$ W; that upon a unit of *r s* is equal to $\frac{1}{10}$ W. Hence, by Law I. the friction of a unit of *m n* is to that of a unit of *r s* as $\frac{1}{2}$ to $\frac{1}{10}$, that is, as 5 to 1; or the friction of a unit of *m n* equals the friction of 5 units of *r s*. Therefore, the friction of the 2 units of *m n* = friction of the 10 units of *r s*.

It must be observed that this law ceases to apply when the extent of rubbing surface is so reduced and the incumbent pressure so augmented as to cause a grooving or tearing up of the surface over which the body is moved. Thus, if the narrow surface *m n* be made greatly narrower and be pressed down with great force while it is moved along horizontally, it will indent and abrade the underlying surface as it passes, and thus bring into play a resisting force much greater than simple friction.

On the other hand, when the pressure is very slight and the extent of the surface very great, the law is again but imperfectly applicable. For, in this case, the adhesion between the two surfaces, which is quite insignificant, compared to the friction excited by ordinary pressures, acquires greater comparative importance on account of the feebleness of the force of friction when the pressure is so slight. It is moreover positively increased in proportion to the surface in contact. Thus, therefore, in these conditions the resistance is not independent of the extent of rubbing surface, but sensibly increased by any great augmentation of surface.

489. Law III. — THE FRICTION IS INDEPENDENDENT OF THE VELOCITY WITH WHICH ONE SURFACE IS MOVED UPON ANOTHER.

First method of Proof. — This law may be proved by using the horizontal table (fig. 250) and applying a moving power P considerably greater than the friction. On marking the distances passed over by W in 1, 2, 3, &c. seconds, they are seen to be in the proportion of 1, 4, 9, &c., that is, proportional to the squares of the times in which they are described. The motion of W is, therefore, *uniformly accelerated.* But to produce such a motion the accelerating force must be of *uniform intensity* (293).

Let A denote the accelerating force, P the applied power, and F the friction; then, as the accelerating force is the excess of the applied power above the friction, we have $A = P - F$, and therefore $F = P - A$. But P the applied power remains unchanged during the experiment, and we have just seen that A, the accelerating force, is constant. Hence, their difference, that is, F the friction, must be of constant intensity through all the stages of the motion.

Second method of Proof. — A similar proof is obtained by placing the body W on an inclined plane (fig. 251), so adjusted as to cause the body freely to descend. The spaces described in 1, 2, 3, &c. seconds counted from the beginning of the motion will be found, as in the preceding case, to have the proportion of 1, ,4, 9, &c., that is, the proportion of the squares of the times. The motion down the plane, therefore, is uniformly accelerated, and the accelerating force must be constant. But this force is obviously the excess of the friction over the force down the plane. Denoting the latter by P, we have as before $A = P - F$, and therefore $F = P - A$. Now, by (262) we have $P : W = A\,C : A\,B$. Hence, $P = W \cdot \frac{AC}{A\,B}$. But during the experiment W and $A\,C$ and $A\,B$ all remain unchanged, therefore P is constant, and, as we have proved that A is constant, it follows that F the friction remains of the same intensity throughout the motion.

Third method of Proof.—A third, and perhaps a more perfect method, consists in using the horizontal table, (fig. 250) and the spring balance. By a suitable adjustment of the motive machinery, the weight W is drawn along the table with a uniform motion. The tension of the spring balance is noted, and the velocity of the traction is now gradually increased to double, triple, &c. of the original speed, and maintained for a while uniform at these higher rates. The spring balance is seen to preserve the tension produced by the first motion perfectly unchanged, although in the course of the experiment the velocity may have been increased twenty fold. But we have already seen that this tension is equal to the friction, and measures its intensity in pounds and ounces. We therefore conclude that the friction is unaffected by the velocity.

490. Since the friction is always opposed to the motion of the sliding mass, and remains of the same intensity whatever be the velocity, as long as the pressure continues the same, we infer that *Friction is a constant retarding force.* Hence, a body set in motion on the horizontal surface $A\,B$ (fig. 251), with a given initial velocity, will proceed with a uniformly retarded motion. Let V denote the initial velocity, and let f denote in feet per second the decrement of velocity due to the resistance of friction. Then, the decrement in t seconds will be $f\,t$, and the velocity remaining at the end of t seconds will be $V - f\,t$. So the space described in t seconds will be computed by the formula for uniformly retarded motions, viz. $S = Vt - \frac{1}{2} f t^2$. It is obvious that if the friction were equal in force to the weight of the body, then f would be equal to g, and the motion of the body on the hori-

zontal plane $A\ B$ would correspond in all respects, except direction, with that of a body shot vertically upwards in a vacuum with the same initial velocity.

491. Of the Coefficient of Friction.

This is the number which represents in each case the ratio of the friction to the weight or other pressure acting at right angles to the rubbing surface. Thus, if the weight or pressure be 10 units, and the friction be found equal to 2 of the same units, then $\frac{2}{10} = \frac{1}{5} = 0\ .\ 2$ is the coefficient of friction.

Denoting the weight or normal pressure by N, the corresponding friction by F, and the coefficient of friction by C, then $\frac{F}{N} = C$. If the normal pressure be unity, as 1 lb. or 1 ounce, this expression becomes $C = F$, that is, *the coefficient of friction is the number which represents the friction of the given surfaces when the pressure is unity.* In the numerical example above given, the Friction being 2 where the Pressure is 10, the Friction will be $\frac{2}{10}$ where the Pressure is 1. That is, $0\ .\ 2$ is the friction when the pressure is unity.

From the expression $\frac{F}{N} = C$, we deduce $F = C\ .\ N$. That is, in a given case knowing the coefficient of friction and the normal pressure, we compute the amount of the friction by multiplying the pressure by this coefficient. This is equivalent to saying that *the whole friction is computed by multiplying the number of pounds or other units in the pressure by the friction proper to one of these units.*

Example. — According to Morin, the mean value of the coefficient of friction for Iron rubbing on Iron is $0\ .\ 44$, that is, for 1 lb. pressure the friction is $\frac{44}{100}$ of 1 lb. Suppose one metallic surface to slide on another when pressed against it with a force of 200 lbs. What is the amount of the friction? Here $N = 200$ and $C = 0\ .\ 44$. Hence, $F = C\ .\ N = 200 \times 0\ .\ 44 = 88$ pounds.

492. From what has been said, it is clear that the coefficient of friction is not varied either by a change of the normal pressure of the rubbing surfaces, or of their extent, within certain bounds, or of the velocity of the motion. But each distinct kind of surface as elm, oak, iron, brass, and each degree of polish or smoothness has a coefficient of fric-

tion proper to itself. So, also, the interposition of an unguent between the surfaces gives rise to a new coefficient depending on the nature of the unguent and the amount applied. Thus, iron upon iron without unguent gives for the coefficient 0.44; when the surfaces are slightly unctuous it becomes 0.18, and when they are separated by a layer of unguent it is reduced to 0.08, between $\frac{1}{5}$ and $\frac{1}{6}$th part of its value when the surfaces are dry.

493. It should be borne in mind, that although the coefficient is not varied by a change of pressure so long as the state of the surfaces remains the same, yet as an increase of the pressure beyond a certain limit will expel a great part of the unguent from between the surfaces, this increase will have the effect of changing the coefficient of friction from that proper to an interposed layer of the unguent to that which is applicable to merely unctuous surfaces. Thus in the case of iron upon iron, while at pressures under the limit, the coefficient is 0.08, an excess of pressure, has the effect of raising its value to 0.18.

494. DETERMINATION OF THE COEFFICIENT OF FRICTION. — It will be apparent from the above, that a knowledge of the coefficient of friction proper to each of the materials used in mechanical structures and for road surfaces, as well with as without unguents, is essential in all practical inquiries respecting the efficiency of machinery and locomotive powers. The following are the methods used for determining the coefficient of friction.

First method. — By the horizontal table (fig. 250). In successive experiments, plates or sheets of different materials, as for example, different woods and metals, are fastened to the bottom of the moveable mass W. Using either the method with the scale-pan and weights, or the attached mechanism and spring balance, we ascertain in each case the amount of the friction. Knowing the weight or normal pressure N upon the lower surface, we divide the friction by this pressure, and thus find the coefficient of friction in each case. $C = \frac{F}{N}$.

Second method. — By the inclined plane. Let the body W be placed upon the inclined plane $A\ B$ (fig. 252), and the plane be gradually raised until W begins to move. In this adjustment the friction may be considered as equal to the force down the plane. Let P denote the force down the plane, F the friction, C the coefficient of friction, and N the normal pressure upon the plane. Then since $P = F$ we have $P = C.N$.

But in the triangle of forces $W\,G\,H$, the line $W\,H$ represents the normal pressure and $G\,H$ the force P.

Hence $P : N = G\,H : W\,H$. But by similar triangles this last ratio is equal to the ratio of $A\,C$ to $B\,C$.

Therefore $P : N = A\,C : B\,C$. Hence $P = \frac{A\,C.N.}{B\,C}$

That is, $C.N = \frac{A\,C.N}{B\,C}$, and therefore $C = \frac{A\,C}{B\,C}$.

That is, *the coefficient of friction is found by dividing the height by the base of the inclined plane, when adjusted to the angle at which motion begins.*

495. The angle φ at which motion begins is called the *angle of friction.* Since $B\,C : A\,C = 1 :$ tangent φ, we have tangent $\varphi = \frac{A\,C}{B\,C} = C$; that is, *the coefficient of friction is equal to the tangent of the angle of friction.*

Iron upon iron without unguent gives $\varphi = 23° 45'$ and tangent $\varphi = C = 0.44$
" " " slightly unctuous " $\varphi = 10° 13'$ " " $\varphi = C = 0.18$
" " " with layer of unguent, $\varphi = 4° 45'$ " " $\varphi = C = 0.08$.

496. If $M\,N$ (fig. 253) be a surface which is pressed obliquely against the plane $H\,K$ with a given force $Q\,P$, it is evident that in proportion as $Q\,P$ approaches the normal $S\,P$, the normal element $r\,P$ increases, and the element $q\,P$ parallel to the surface diminishes. Hence the friction increases, and the force tending to slide $M\,N$ upon $H\,K$ diminishes. There is, therefore, a certain inclination of $Q\,P$ to the normal $S\,P$, at which the sliding force $q\,P$ is exactly equal to the friction. When the impressed force $Q\,P$ makes with the normal a less angle than this, there can be no motion. When it makes a greater angle, $q\,P$ exceeds the friction, and $M\,N$ is set in motion in the direction $P\,H$. This angle is called *the limiting angle of resistance.*

497. *The limiting angle is the same, whatever be the mag-*

nitude of the oblique force. — For if we suppose $Q\,P$ to be doubled, we shall have a double normal element $r\,Q$, and, therefore, a double friction, and we shall have a double parallel element $q\,P$ to balance this friction. In whatever proportion we increase or diminish the impressed force, the result will evidently be the same.

Hence the proposition, that *whenever any single force or resultant of forces operates so as to press one surface against another, it will not cause the one surface to move upon the other, unless the angle included between this force and the normal be greater than the limiting angle.*

498. TO DETERMINE THE VALUE OF THE LIMITING ANGLE. — Let x (fig. 254) be the limiting angle for the surface $M\,N$, and let $Q\,P$ denote the oblique force. Then $R\,P$ will be the normal force, and $Q\,R$ the parallel force. At the limiting angle we have $Q\,R = F$, the friction. But by (491) $F = C\,.\,N$. Hence by substitution, $Q\,R = C\,.\,R\,P$ and $\frac{Q\,R}{R\,P} = C =$ tangent ϕ. But $R\,P : Q\,R = 1 :$ tangent x. Therefore tangent $x = \frac{Q\,R}{R\,P}$. Hence tangent $x =$ tangent ϕ and $x = \phi$. That is, *the limiting angle of resistance is equal to the angle of friction.*

Example. — Suppose a surface of iron to be pressed obliquely against another surface of iron, in what direction must the pressure act in order to cause the motion? Here the coefficient of friction is 0.44. But this is equal to tangent φ. Whence by the tables we find $\phi = 23° 45'$. That is, if the applied force make with the normal any angle greater than 23° 45′, it will cause motion.

499. OF THE CONE OF RESISTANCE, OR FRICTION. — Let $Q\,P\,R$ (fig. 255) be the limiting angle or angle of friction proper to the surface $M\,N$, and suppose $Q\,R$ to revolve around the normal $R\,P$, preserving always the same inclination to it. The conical surface thus described is called the cone of resistance or friction.

From what has been said above, it is plain that a force applied in the direction of $Q\,P$, $S\,P$, or any line *lying in the conical surface,* will give rise to a friction exactly equal to the sliding force; that when the force is applied in a line *S P external to the cone*, the sliding force will exceed the friction; and that when applied in a line *S P within the cone*, the sliding force will fall short of the friction, and there will be no motion.

500. OF THE MOST ADVANTAGEOUS ANGLE OF TRACTION. — Let G be the centre of gravity of the body $M N$ (fig. 256) and let φ be the limiting angle. Suppose a force to be applied in the direction $G T$, and take $G W$ and $G a$ to represent the weight of the body and this force respectively. Complete the parallelogram $a W$. It is evident that $G b$ denotes the direction and value of the force by which the surface $M N$ is 2 6
pressed against $H K$. If now $G b$ be in the direction of $Q G$, prolonged it will make with $M N$ the limiting angle, and the surface $M N$, although not moved, will be in the state bordering on motion, so that a very slight addition to the force $G a$ will suffice to cause motion. If now we vary the direction of $G T$, we will find $G a$ or its equal $W b$ to vary. The tractional force necessary to balance the friction will, in each direction in which it may be applied, be measured by a line drawn from W to the fixed line $G V$, and parallel to the tractional force. But the shortest line which can be drawn from W to $G V$ is the perpendicular $W x$; therefore the tractional force is least when applied at right angles to the line $Q G V$. But (fig. 257) when $G T$ is perpendicular to $Q V$, we have $i = 90° - R G T$ and $a = 90° - R G T$. Therefore $a = \varphi$. That is, the angle of traction at which the least force will be required to bring the surface into a state verging on motion, or what is practically the same thing, at which it will just suffice to give it motion, is equal to the limiting angle, or angle of friction.

501. Since the angle of friction is equal to the angle of elevation of an inclined plane, down which the body would just begin to slide by its own weight, we have the rule, *that the direction of most advantageous draught on a given surface, is a line making the same angle with the surface, which the surface itself, if in an inclined position, should make with the horizon, in order to cause the body to begin to move upon it.*

OF AXLE FRICTION.

502. When a wheel embraces loosely a fixed cylindrical axle (fig. 258), or when an axle revolves in cylindrical boxes (fig. 259), the moving surface slides on that which is fixed, and thus gives rise to a peculiar kind of sliding friction. By numerous experiments it has been proved that this, like the sliding friction, of which we have just been treating, is directly proportional to the normal pressure.

503. TO FIND THE COEFFICIENT OF AXLE FRICTION. — Owing to the new conditions in which the rubbing surfaces are placed, the coefficient of axle friction differs materially with surfaces of the same nature from that of the sliding friction above considered. Hence the importance of special experiments for its determination. Let the pulley $A M B$ (fig. 258) rest on the axle $P D Q$. The bearing surface will be at and near D. From a cord passing over the pulley suspend the two equal weights W and W. Then by degrees augment the weight on one side until the excess w is barely sufficient to overcome the friction at D, and cause the pulley to move. We may now compute C, the coefficient of friction proper to this case, in the following manner.

Let N denote the whole pressure acting at D, consisting of the weight of the pulley and the sum of the weights suspended by the cord. Also let R denote the radius of the pulley and r that of the axle, and let F be the friction at D, that is, in this case, the entire friction of the axle. As the friction is proportional to the pressure, we have $F = C . N$. But when the weight w is just sufficient to cause motion, it may, without serious error, be assumed as in equilibrium with the friction. Hence the moment of $w =$ the moment of F; that is, $F . r = w R$; or by substitution $C . N . r = w . R$; therefore $C = \frac{w R}{N r}$.

When, as in (fig. 259), the axle is fixed in the pulley and rests in a cylindrical box in which it may revolve, the resultant pressure will act at the lowest point of the axle, and the equation of moments will be the same as above, and, therefore, the coefficient of friction the same.

504. The general result of these experiments presents for

the coefficient of axle friction a value less than for common sliding friction. Thus for iron and other metals, separated by a layer of unguent, this number is 0.05 instead of 0.08; that is, only ⅝ lbs. of the coefficient for the same surfaces in common sliding friction.

505. When the weight w is sufficient to give continuous motion to the pulley, this motion is found to be uniformly accelerated; whence we infer that the accelerating force is constant. But this is the excess of the force which w exerts over the resistance of the friction. As w is constant, and this excess is constant, it follows that the friction is constant. In other words, *axle friction, like common sliding friction, is independent of the velocity.*

OF ROLLING FRICTION.

506. The Laws of Rolling Friction as deduced from the experiments of Coulomb and others, are as follows:

Law I. — WITH A GIVEN KIND OF SURFACE AND A GIVEN RADIUS FOR THE CYLINDER OR WHEEL, THE ROLLING FRICTION VARIES DIRECTLY AS THE WEIGHT OR PRESSURE EXERTED BY THE ROLLING MASS UPON THE SURFACE BENEATH.

Proof. — A cylinder, (fig. 260) is placed upon a horizontal table, so as to lie centrally and transversely over a long and narrow opening in the top of the table. A strong and very flexible thread is passed over the cylinder, and its ends carried down through the opening. To these are attached two scale-pans for receiving weights, by means of which any desired pressure may be applied.

Let N denote the entire pressure acting at D, the point of contact of the roller with the table, small weights are now to be added on one side until the roller begins to move. The excess of weight w being noted, the pressure N is increased to $2N$, $3N$, &c., when it will be found that the force necessary to overcome the friction at D becomes $2w$, $3w$, &c., that is, the friction increases as the pressure.

507. Law II. — WITH A GIVEN KIND OF SURFACE AND A GIVEN PRESSURE, ROLLING FRICTION IS INVERSELY PROPORTIONAL TO THE RADIUS OF THE ROLLER.

Proof. — This law is proved by using a series of rollers, having surfaces of the same kind, but whose radii are in proportion of 1, 2, 3, &c. When by adjusting the weights, or by other means, the pressure N is kept invariable, it is found that the forces necessary to give motion to the several rollers are in the proportion of 1, $\frac{1}{2}$, $\frac{1}{3}$, &c. That is, with a roller of double radius the friction is one half; with one of triple radius, it is one third. Let N denote the pressure, and R the radius of the roller, C the coefficient of rolling friction, and F the whole friction, or the weight w necessary to overcome it. Then combining the preceding laws we have $F=\frac{C.N}{R}$. Whence $C=\frac{F.R}{N}$, that is, *the coefficient of rolling friction is found by multiplying the radius by the actual friction, and dividing by the pressure.*

508. Thus far we have regarded the pressure, and therefore the friction as acting only on one side of the roller. But when a roller is interposed between a load and the surface on which it is to be moved, the pressure acts at the top as well as the bottom of the roller, and there is friction as well where the load rests upon the roller as where the roller presses on the surface which supports it.

Let the mass W (fig. 261), supported on rollers, be moved over any level surface in the direction EF. It is evident that as the roller AD revolves through a given arc QA, it pushes the surface MN forward through a space equal to QA. Thus the point A originally over the centre of this roller is at a distance equal to QA in advance of that centre. But meanwhile the roller has advanced on HR in the same direction through a distance $Dd=QA$, and with it has carried its load. Therefore, *the whole space described by the load is equal to twice that described by the roller.*

The force necessary to give motion to W is the sum of the friction at the upper and that at the lower surface. Let C and c be the coefficients of friction for the upper and lower surfaces respectively. Then the former force is equal to $\frac{CW}{R}$, and the latter to $\frac{cW}{R}$. Hence the whole force to be overcome in moving the mass is $\frac{CW}{R}+\frac{cW}{R}$, that is, $(C+c).\frac{W}{R}$.

510. OF WHEEL VEHICLES. — When wheels are interposed between the burden and the road, and are locked or prevented from rotating, the motion of the vehicle is resisted by sliding friction acting at the surface of contact with the road. Let C denote the coefficient of this sliding friction, and let L be the weight of the load and carriage, and w that of the wheels. Then if F denote the force necessary to move the vehicle, we have $F = (L + w)\,C$.

If now the wheels be unlocked and free to revolve, there will arise sliding friction at Q, the circumference of the axle, and rolling friction at P, where the wheel touches the road (fig. 262). The former acts with the leverage $C\,Q$, the radius of the axle; the latter with the leverage $C\,P$, the radius of the wheel. Let C_1 be the coefficient of the axle friction at Q; then the whole friction at Q denoted by $Q\,S$ will be equal to $C_1\,L$. Let F_1 denote the force which is just capable of giving motion to the wheel. By the friction at P, an equal and opposite force $P\,R$ is brought into action, which is barely able to overcome the friction at the axle. We may, therefore, assume these forces in equilibrium. Denoting by R and r the radii of the wheel and its axle, we have $F_1\,R = C_1\,L\,r$ and $F_1 = C_1\,L\,\frac{r}{R}$.

511. If in two vehicles the axle surfaces and loads be the same, we have $C_1\,L$ the same in both, and therefore F_1 varies as $\frac{r}{R}$. That is, *the forces necessary to move equal loads on the same surfaces, are proportional to the radii of the axles divided by the radii of the wheels.* Thus an axle of double radius requires double force. A wheel of double radius only half the force.

Example. — A loaded railway car weighs, without the wheels, 2400 lbs., and the wheels weigh 100 lbs. The radius of each wheel $= R = 1.5$ feet, and that of each axle $= r = 0.05$ feet. The axle surfaces are of greased Iron, making $C = 0.08$. We have, therefore, $F = 0.08 \times 2400 \times \frac{0.05}{1.5} = 6.4$ lbs. Thus a force of $6\frac{4}{10}$ lbs. is sufficient to put this car in motion, when its wheels are free to rotate.

Suppose the wheels to be locked, we have as above (510),

$F = C\ (b+w)$. But here the sliding friction is on the rail, and its coefficient $C = 0.2$ nearly, therefore, $F = 0.2 \times (2400 + 100) = 0.2 \times 2500 = 500$ lbs. Hence, $F_1 : F = 6.4 : 500 = 1 : 75$. *It thus appears, that about 75 times as great a force would be required to give motion to the car with the wheels locked as when they are free to revolve.*

512. In order to cause the rotation and rolling of the wheels, the amount of sliding friction on the road surface at *P* (fig. 263) must be sufficient to overcome the axle friction at *Q*. If the surface be very slippery, as when the road is covered with ice, the friction excited at *P* by the forward traction of the car is not sufficient to overcome the friction at *Q*, and the consequence is that the wheels slide without revolving. The same result occurs in the ordinary condition of the road surface, provided the axle friction is so much increased as to render the force at *P* incapable of revolving the wheel.

513. In the case above considered, the force which gives motion to the vehicle is a traction applied directly to it, and the rotation of the wheels is the result of the frictional resistance at the road brought into play by the forward motion of the vehicle and its wheels.

But there is another mode of action of the propelling force, viz., that in which this force is applied directly to two or more of the wheels causing them to revolve, and, as a result of this rotation and the resistance at the road way, causing them to roll and the vehicle to advance.

514. It is in this way that the force acts in the locomotive engine and in the ordinary hand-car worked by a crank. Let *S P* (fig. 264) represent one of the driving wheels of a locomotive, *B A* being the rod connecting it with the engine. As *B A* moves backward and forwards, its extremity *A*, attached by a joint to the surface of the wheel, is made to describe the circle *A a*, and carries the wheel around in the same direction. If there were no friction at *P*, the periphery of the wheel would slide past this point on the road, and there would be rotation without progressive motion. The same result will take place, when the force with which the rotation carries the point *P* of the wheel backwards is greater than the sliding friction at that point. For then, *P* will be urged to slide or revolve past the same fixed point in the

road by the excess of this rotating force over the frictional resistance.

The friction at *P*, by preventing a sliding motion, renders this point at each moment a fulcrum or centre of motion. The action of *B A* forward drives the centre *C* forward, and thus meets and overcomes the combined friction of the hind wheels of the locomotive and the resistance of the train of attached cars, which may be regarded as a single force acting in the direction *x y*. If this backward force be very great, as in the case of a heavy train of cars, it will require a correspondingly great friction at *P* to enable the force of the driving rod *B A* to overcome it and produce a forward motion of the wheel. When the force in the direction *x y* is in excess, the resistance at *P* will be unable to withstand its action, the fulcrum will give way, and the wheel will slide past the point *P*, revolving, but making no progress.

CHAPTER XXV.

OF THE ELASTICITY AND STRENGTH OF SOLID MATERIALS.

515. OBSERVATION shows that all solid bodies are capable of being compressed by forces applied in certain directions, and extended by forces applied in the opposite directions; and, further, that on the removal of the compressing or extending force, the body tends to recover its original form and dimensions. This power of restoration is what is called Elasticity. Familiar examples are presented in the vibration of a stretched cord or of a bell, or in the recoiling force by which a bent spring returns to its original form and position. In these cases the restoration of the body to its previous shape is perfect, the displaced particles may be considered as having returned to precisely the same relative position as before the disturbance, and the experiment may be repeated with the same disturbing forces again and again without giving rise to any permanent change in the shape and dimensions of the mass. Experience proves, however, that this com-

plete restoration does not take place even in cases like these, when the change of form first produced is very great. After applying an excessive force, we find that, on coming to rest, the cord has been permanently stretched, and the bell and spring have been permanently bent, or otherwise changed as to the arrangement of their parts.

516. For each kind of material there is a certain limit to which the particles may be displaced without losing the power of restoring themselves perfectly to their original position. So long as the disturbing force, whether a compression or an extension, does not exceed this limit, the body is able to recover its original form completely, but when the displacement is carried beyond the limit, it is able only in part to regain its former shape, and thus suffers a permanent change of dimensions. The limits of elasticity of different materials differ greatly as to their extent. Thus, while a long blade of highly tempered steel may be bent almost double, and yet be able to regain its original straightness, a slip of lead, of the same dimensions when similarly bent, will scarcely recoil in a perceptible degree. So a ball of gum elastic indented by pressure will quickly recover its shape, while one of wax or wood will remain permanently flattened. But although bodies differ thus greatly in their limits of elasticity, it will presently be proved that each within its appropriate limit must be regarded as perfectly elastic.

517. When, after the removal of the disturbing force, the restoration of form is perfect, we must regard the internal force, which effects the restoration, as equal to that which produces the change. Hence, within the limits of elasticity, the restoring force is equal to the disturbing force, and increases with it until the elastic limit is reached. At the limit this force has the greatest intensity possible for the given material. If the disturbing force be still further increased, the restoring force will no longer be able to effect a complete recovery.

Where the disturbing force is a *pressure*, it is in practice generally kept within the elastic limits of the materials sustaining it, and therefore the restoring force which is brought into play is equal to it. But, in cases of *impact or collision* the displacement at the point struck in almost every instance exceeds the limits of elasticity, and there is only a partial restoration of form. If D be used to denote the disturbing force, and R the force of restoration, the ratio of the latter to the former represented by the fraction $\frac{R}{D}$ is called the *Coefficient of Elasticity*. This is usually denoted by e. Thus, $e = \frac{R}{D}$. Within the limits of elasticity $R = D$, and therefore $e = 1$.

518. The limits of elasticity of different materials, may be compared experimentally by either of the following methods.

First, *by longitudinal tension.*

Second, *by transverse deflection.*

Third, *by twisting or torsion.*

Of these, the last furnishes the most striking results, and will best serve to illustrate what has been said above. The other modes of experimenting on this subject will be noticed under a succeeding head.

519. COMPARISON OF THE LIMITS OF ELASTICITY BY TORSION.—Let the material of which trial is to be made be formed into a cylindrical rod or wire of given length and diameter. Fastening it by a clamp or vice at the upper end, so that it cannot turn, let it hang vertically over the centre of a graduated circle (fig. 265). Having attached to its lower end a horizontal index, and, if it be necessary, a weight, we proceed as follows. We turn the index and thus twist the wire, and then allow it to recoil. After some vibrations the index will come to rest, and if it be found to point to the same mark on the scale as at the beginning, we know that the particles of the wire have regained exactly their first positions. We now deflect the index through a larger angle, and thus proceed until we attain the limit beyond which the restoration of the index to the original mark fails to be perfect. The number of degrees through which the index has been revolved, or the wire twisted, at this time, is set down for each material, and the numbers thus found are evidently in the same proportion as the limits of elasticity of the several rods or wires which have been used.

By proceeding in this way, it will be found that a slender wire of steel will bear several revolutions without exceeding its elastic limits. An equal wire of brass will admit of somewhat fewer, and one of copper will bear still less torsion, and one of lead will not admit of being twisted through more than a few degrees without losing its power of perfect restoration. But within this limit of torsion, the return of the index is just as entire and absolute with the lead as with the steel. By making similar experiments with rods of the softest woods, of clay, wax, and other substances usually regarded as being without elasticity, it has been proved that each has its proper elastic limits, within which its power of restoration is just as perfect as that of steel. From numerous experiments, all leading to the same result, we are authorized to infer that, while bodies differ greatly as regards their limits of elasticity, they are all perfectly elastic within these limits.

520. OF THE TORSION PENDULUM. — The simple apparatus above described becomes, when properly constructed, a very delicate and accurate means of measuring forces of small intensity, and it is thus used under the name of the Torsion Pendulum or Torsion Balance. To understand its application, it is necessary first to consider the molecular forces excited by the twisting of the wire; and, secondly, to determine the laws which govern its vibrations.

521. *First.* — OF THE MOLECULAR FORCES EXCITED BY TORSION. — When a cylindrical rod or wire, fixed at one end, is twisted by a force applied at the other, all parts of it are twisted to the same degree. The wire may be regarded as made up of an indefinite number of circular layers laid upon one another, each of them having the thickness of a single particle.

Let *A B* (fig. 266) represent such a structure on a magnified scale. Suppose each elementary layer to contain only six particles, and that the length of the wire comprises 12 such layers, excluding that at the top. If the lower layer, of which the particles *a b c* are seen in the figure, be turned in the direction of the arrow, they will be removed laterally to a greater distance from those immediately over and nearest them, and will attract them so as to compel them to follow in the same direction. These, in their turn, will cause a similar lateral displacement of the third layer, and thus by the sliding of layer upon layer, the lateral displacement will be extended to the top of the wire. This movement will continue until the resistance opposed by the lateral tension thus excited becomes equal to the applied force of torsion, when the two will be in equilibrium. The lateral tension and displacement must be the same between every two contiguous layers. For if it were not so, there would be motion in the direction of the greater tension, and a consequent change of the tension bringing them to a state of equality.

Thus, in the twisted state of the wire, the particles will be arranged as in (fig. 268). Suppose the lower end of the wire, or the index attached to it, to be turned around through an entire circle, the amount of displacement at each of the 12 joints 1, 2, 3, &c., would be $\frac{360}{12} = 30$ degrees. If, instead of 12, we imagine 12,000,000 of such layers of particles, the amount of lateral displacement proper to each would be the fraction $\frac{360}{12000000}$ in angular measurement, or in actual distance, the $\frac{1}{12000000}$th part of the circumference of the wire. The circumference of bell wire is about $\frac{1}{10}$th of an inch. With a length equal to 12,000,000 of particles, this would give a displacement of each amounting to the $\frac{1}{120000000}$th of an inch; and yet this almost infinitely small change would be marked by the motion of the index through an entire circle.

LAWS OF TORSION.

The following laws apply to the torsion pendulum, when the displacement of the particles does not exceed the limits of elasticity.

522. Law I. — *The angle of torsion, that is, the amount of twist given to a rod or wire, is exactly proportional to the force applied.*

This may be proved experimentally, as follows. Let the clamp *B*, by which the wire is secured at top, be a conical or cylindrical pin, fitted to a socket in the centre of a graduated circle at that end of the instrument, so that the pin may be turned round, and carry with it an index which is fastened upon it a little above the graduated surface (fig. 268). A slender rod fastened to the lower end of the wire, in a horizontal position, has a point of steel projecting down-

wards from its centre, and entering a small cavity
in a piece of steel or agate placed in the centre of th
disc. While the wire remains free to revolve, it is
that by turning the index B through any given an
wire and rod CD will be carried around through a
angle. If, however, CD be held fast while the ind
turned, the wire will be twisted to an amount exactl
to the angular revolution of the index. A flexible
attached at C in a horizontal direction, and at right
to CD, is carried over a pulley and supports a ligh
pan intended to receive weights. A pin or detent
near C restrains the bar CD from moving in the di
CF under the action of the weight.

On placing a small weight w in the scale, the bar
pressed strongly against the pin. We now turn the i
in the direction of the upper arrow, and thus twist t
until the force of torsion excited in the wire is just
balance the weight applied at C. Marking the angle
sion ab at which this occurs, we next add weights e
$2w$, $3w$, $4w$, &c., moving the index B in each cas
equilibrium is restored. It will be found that the a
torsion corresponding to these weights will be $2a$
$4ab$, &c. In other words, the angle of torsion is
tional to the intensity of the twisting force.

Since, in this experiment, the molecular force exci
the torsional displacement is always in equilibrium w
weight applied, it must be proportional to the weight.
necting this with what has just been proved, it follo
this molecular force is proportional to the angle of tor

523. Law II. — *The times of vibration of the torsi*
dulum are equal, whether the excursions be great or sm

This is readily proved by adjusting a metronome o
variable pendulum so as to vibrate at the same rate w
torsion pendulum, when its vibrations are confined t
arcs. If it be then caused to swing in double or tripl
it will be seen that the time of vibration remains the

This law may be proved independently of expe
for it is a dynamical result of Law I., as will appea
the following reasoning. The forces acting upon the
lum when at any distance mc, nc, from the point

are evidently the molecular tensions due to the amount of twist represented by *m c*, *n c*. But by Law I. these forces are proportional to *m c*, *n c*, the angles of torsion. Thus the force accelerating the pendulum towards *c* is proportional, at different points of the arc of vibration, to the distance to be passed over. Hence, as in the case of the cycloidal pendulum, the times of vibration will be equal, whether the arcs be great or small.

524. Law III. — *The time of vibration of the torsion pendulum is directly proportional to the square root of the length of the wire.*

This law is proved experimentally by clamping the wire successively at different points, and thus reducing its length to that of the part which lies below the clamp. Taking successive lengths in the proportion of 1, 4, 9, 16, &c., we find the corresponding times of vibration to be in the proportion of 1, 2, 3, 4, &c., that is, $\sqrt{1}$, $\sqrt{4}$, $\sqrt{9}$, $\sqrt{16}$, &c.

525. Law IV. — *The time of vibration of the torsion pendulum, is directly proportional to the square root of the weight suspended at its lower end, this weight being in the form of a cylinder or prism of uniform horizontal section.*

Making use of a series of short cylindrical weights of equal diameter, we add them in succession so as to have the series 1, 4, 9, 16, &c. The corresponding times of vibration will be found to be 1, 2, 3, 4, &c., the square roots of the former.

OF THE TENSION DUE TO LONGITUDINAL FORCES.

526. Observation proves that when a bar *A B*, (fig. 269) is loaded at its lower end with a weight *W*, its length is thereby increased; and, again, that when the weight rests upon the upper end of the bar, the lower extremity being fixed, its length is diminished, (fig. 270).

As the bar, after undergoing the extension proper to the action of the weight, ceases to be further extended by it, we must admit that in the new position of the particles of the bar arising from its extension, a force is excited capable of holding the weight in equilibrium. While the particles are at their natural distances they do not support the weight This is evident from the fact, that the weight descend a small distance immediately after it is applied, and thus extends the beam, that is, draws the particles further from each other. It therefore appears, that the removal of the particles from each other enables them jointly to support the weight. In the second case the weight is not sustained while the particles remain at their natural distances. This is shown by the descent of the weight when just applied, and the consequent compression of the beam. In this case, therefore, it appears that the approximation of the particles enables them jointly to support the weight.

We therefore infer that, at their natural distances, the particles of beams and other solid masses may be regarded as exerting no forces, or, what is equivalent to this, they may be regarded as exerting forces which are mutually balanced; but the moment they are separated or brought nearer, energetic forces are called into action. It is moreover evident, that when the particles are removed further from one another, the force excited is an attraction tending to restore them to their previous positions; and that when they are pushed nearer together, the force excited is a repulsion tending also to reinstate them. We conclude, then, that *bars, beams and solid strictures generally, acted on by weights or other forces, are enabled to sustain their pressure, not by any forces already acting among their particles, but in virtue of the attractions and repulsions which are excited among the particles, according as they are drawn further apart or forced nearer to one another.*

527. In the case of bars and beams, such as *A B*, (figs. 269 and 270) the extensions or compressions produced by *W* are shared equally by all portions of the length of the beam. For if at any part of the beam there should be a greater extension or compression than at the adjoining parts, this would excite a stronger attractive or repulsive force between the particles there situated than in the rest of the beam. Hence, they would approach, in the one case, and recede in the other, until their forces became equal with those of the other particles of the beam, and this could only happen when their distances became equal.

528. We may infer from this, that with a given load a beam of double or triple length will sustain a double or triple

extension when the load is suspended below, and a double or triple compression, when it rests upon the beam. In other words, the total increase or diminution of length under a given pressure is proportional to the length of the beam. It follows, from this, that the liability of a beam to be ruptured by a force of extension, is not influenced by any variations in its length further than is due to its weight. In the case of vertical suspension, this weight should always be included as a part of the load to which the beam is subjected.

529. OF THE LIMITS OF ELASTICITY OF BEAMS AND BARS STRETCHED OR COMPRESSED IN THE DIRECTION OF THEIR LENGTH. — From numerous experiments made upon different metallic wires, and on rods of metal and wood, the limits of elasticity of these substances have been inferred with considerable precision. The mode of experimenting is simply to load the wire or rod with a successively increasing pressure, until we find it beginning to undergo permanent change of length. The greatest extension short of this change is the limit of elasticity. Thus, according to Barlow, a bar of *malleable railway iron*, of an inch square in the section, is extended $\frac{1}{10000}$th of its length by a direct strain of one ton weight, and when the weight is withdrawn the bar recovers precisely its original length. Strained by a force of 2, 3, 4 tons, it is stretched 2, 3, 4 times as much, and still is able to regain its original length when the torsion is removed. This continues to be the case until the load amounts to 10 tons, when the extension is $\frac{1}{1000}$ of the length of the bar; with a load greater than this the bar regains its former length only in part, and remains permanently lengthened. Thus, the fraction $\frac{1}{1000}$ denotes the limit of elasticity of this kind of iron.

In the same way it was found, that an equal bar of *cast-iron* reaches its limit under a pressure of 15,300 lbs. $= 6\,^{8}$ tons, and the extension amounts to $\frac{1}{1204}$th of the length of the bar. Thus, $\frac{1}{1204}$ denotes the elastic limit of cast-iron. According to the same authority, a bar of *oak* of the same section will bear a weight of 3960 lbs., and be extended $\frac{1}{480}$ of its length, without having its elasticity impaired, and a bar of yellow pine, of the standard size, will bear 3900 pounds, and be stretched $\frac{1}{414}$th, still retaining the power of restoration.

Comparatively few direct experiments have been made to ascertain the limits to which compression may be carried without a permanent alteration of length, but these limits may in many cases be inferred from the flexure produced by transverse forces; and the existence of such limits is clearly shown by the permanent alteration produced by excessive compressing forces, and the restoration of form which follows when the forces have been of moderate intensity.

530. *Within the limits of elasticity the amount of extension or compression is proportional to the extending or compressing force.*

The observations of Barlow, above referred to, serve to illustrate this law in the case of malleable iron. A bar of 1 inch iron loaded successively with 1, 2, &c. tons was extended as in the following table:

Stretching force in tons,	1	2	3	&c. to	10
Extension, ...	$\frac{1}{10.000}$	$\frac{2}{10.000}$	$\frac{3}{10.000}$	&c. to	$\frac{10}{10.000}$

Similar results are obtained with cast-iron and other materials, the amount of extension differing in each, but not in proportion to the extending force. This principle appears to hold good also in the case of compressions, when the displacement is very small, but with most materials it ceases to apply earlier than in the case of extension. Hence, even within the limits of elasticity, the compressions begin to increase less rapidly than the compressing forces.

When, either by extension or compression, the limits of elasticity have been passed, the particles take a new set, and it is generally observed that in the new arrangement they oppose less resistance to the tensile force than at first. Thus, an elastic cord violently stretched, becomes weaker, and a column excessively loaded at the top becomes crippled, and less able to bear a great pressure than before this alteration of its structure.

531. *The amount of extension produced by a given force, and of compression produced by the same force when acting the opposite way, are not in general equal.*

In some cases, especially when the displacements are small compared with what the materials will bear without injury to their elasticity it is found that a given force will just as much of extension when applied in one direction as of compression when reversed. But this equality cannot be assumed as a general law. Observation has shown that, with most materials, the extension is greater than the compression, when the force applied is the same, although in a few instances an opposite relation obtains. In cast-iron, according to Tredgold, the extensions and compressions are equal.

OF THE TENSIONS EXCITED BY TRANSVERSE FORCES.

632. When a beam is fixed at one end in a horizontal position, and a force is applied transversely at the other, (fig. 271), this force causes the descent of the outer end. If the beam be of wood or metal, and the force not excessive, the downward movement is soon arrested, and the whole beam exhibits a slight degree of flexure. The upper surface *A a* is bent into the curve *A q*, having a slight convexity upwards and the lower surface *B P* into the curve *B p*, parallel with the former. In this bent condition the upper part of the beam is in a state of extension, and the lower in a state of compression. To show this, experimentally, two narrow slits are made across the beam in its upper and lower surfaces as at *M* and *N*. When the weight or other transverse force is applied, *M* is observed to grow wider and *N* to contract.

633. It is evident that the extension of the fibres or lines of particles in *A q* is greater than in any other part of the beam, and the compression in *B p* greater than any other compression. Receding from *A q* downwards, the extensions become less and less, and similarly from *B p* upwards the compressions regularly diminish. There must, therefore, be some intermediate line of particles which sustains neither extension nor compression. This line of particles *x z* retaining their ordinary distances is called the NEUTRAL AXIS. Taking into account the horizontal breadth of the beam, it is evident that the neutral axes of all the sections like *A B P Q* will have like positions between *A Q* and *B P*, and will thus form a surface, which may be called the neutral surface of the beam.

634. *The position of the neutral axis varies with the nature of the material.*

We have already seen that the compressions and extensions produced by the same force are not, necessarily, or even generally, equal. When the extension of *A Q* exceeds the compression of *B P*, and when the same relation is true of the fibres below *A Q* in the upper part of the beam, as compared with those above *B P* in the lower part, it is obvious that more than one half the beam is in a state of extension, and consequently the neutral axis is placed below the central line of the beam. When the conditions are reversed, this axis will be above the central line. In the case of an equality between the extensions and compressions of the corresponding fibres above and below, the neutral axis will be in the middle of the beam.

It has been stated already, that, in general, materials are more extensible than compressible by the same force. Observation has accordingly shown, that, in nearly all cases, the place of the neutral axis is below the centre. In malleable iron its distance from the upper or more extended surface of a rectangular beam is about ⅘ths of the entire depth. In cast-iron the axis is very near the middle of the beam.

An ingenious mode of finding the position of the neutral axis by direct observation was used by Barlow, in connection with the experiments formerly noticed. A groove was cut in the bar (fig. 272), as shown at *x y*. To this a steel key was fitted accurately, but so as to move with ease. When the strain was on, the key was inserted, it descended until stopped by the narrowing of the groove resulting from the compression towards the lower part of the beam. The point at which it was arrested, marked the beginning of compression, and therefore gave very nearly the position of the neutral axis.

535. *Whatever be the position of the neutral axis, the aggregate of the forces of extension on the one side, is equal to the aggregate of forces of compression on the other.*

If in the beam (fig. 271) the sum of the extending forces were greater than that of the compressing ones, there would remain a force equal to the difference of the two, acting outwards in the direction parallel to *A Q*, and if the compressing forces had the advantage, there would remain a force equal to the difference acting towards the wall. No such action results from the pressure of a weight at *W*, nor indeed could it result consistently with mechanical laws.

OF THE DEFLEXIONS ARISING FROM TRANSVERSE FORCES.

536. We have seen that when a beam is bent by transverse action, one side of it is extended while the other is compressed; and that the deflexion is the joint result of the lengthening of the one side and the shortening of the other. If the extensions and compressions do not pass the limits of elasticity of the given material, as soon as the straining force is removed, the beam recovers its original form.

537. *Within the limits of elasticity, the deflexions are proportional to the weights or other transverse forces producing them.*

This law might be inferred from the fact that, within these limits, the extensions and compressions are proportional to the forces which cause them, and at the same time the deflexions are proportional to the extensions and compressions of which they are the result. But it admits of very easy experimental proof. A bar of metal or wood is placed horizontally, either fixed at one end (fig. 273) or supported on props at both extremities (fig. 274). Weights are attached to the free end of the one or to the middle point of the other, and the depression produced in each case is noted. It will be found, that within the limits of elasticity the deflexions are proportional to the weights applied.

538. Combining the results of experiments on longitudinal tension and on torsion with those just cited, we may lay down the general principle, that *within the limits of elasticity the extent of displacement, whether it be an extension or compression, a torsion or deflexion, is proportional to the force producing it.*

OF THE COMPARATIVE STRENGTH OF BEAMS, ETC., SUBJECTED TO FORCES VARIOUSLY APPLIED.

539. By the strength or tenacity of a beam is meant its power of supporting a weight or other force, either direct or transverse in its action. It is measured by the amount of force which the beam is just able to bear without breaking.

The *direct strength* of a beam is the utmost force which it is able to bear in the direction of its length. The *transverse strength* is the utmost force which it can support when the direction of the force is at right angles to its length.

The direct strength is the same for all beams of the same material and the same sectional area, however various the form of the section. The transverse strength varies according to the figure of the section, the number of supports, the length of the beam, and other particulars.

540. The term *elastic strength* may be used to indicate the power of sustaining a force either direct or transverse, without permanent alteration; and this will be measured by the amount of force which the beam is just able to bear, without being extended or compressed beyond the limits of elasticity.

541. OF DIRECT STRENGTH. — *The direct strength of a beam is found by multiplying the area of the section in square inches by the utmost load in pounds which a beam of the same material, one inch in section, is able to support.* Using A to denote the area of this section in inches, and S the direct strength of the unit beam, we have direct strength $= S A$. Observation shows that a one-inch beam of New England Fir will just support an extending force of 12,000 lbs., one of Pitch Pine 10,500 lbs., and one of Locust 20,580 lbs. Hence the direct strength of New England Fir $= 12{,}000\,A$, of Pitch Pine $= 10{,}500\,A$, of Locust $20{,}500\,A$.

The direct elastic strength is found by multiplying the section by the load in pounds, which the unit beam is able to bear without exceeding its elasticity. Calling A the area and E the elastic strength of the one-inch beam, we have direct elastic strength $= EA$. It has been found that a one-inch bar of New England Fir will bear 3,630 lbs. without injury to its elasticity; a similar bar of malleable iron 17,800. Thus the direct elastic strength is for the former $3{,}630\,A$, and for the latter $17{,}800\,A$.

542. In what precedes, the *weight of the beam* has been purposely omitted; but in practice this can only be done safely where the applied weight or other strain very greatly exceeds the weight of the beam. Considering now the effect of this latter force, it is evident that while the uppermost

section $A B$ (fig. 275) is called upon to support a load equal to W plus the weight of the beam, the section $E F$ sustains a load equal to W plus the weight of so much of the beam only as lies below $E F$; and, finally, the section at $C D$ supports W alone. Hence the tension of $E F$ is less in proportion as it is farther from $A B$. Let us now cut away parts of the beam on each side, leaving a tapering form of such proportions that the sections $N O$, $N_1 O_1$ shall diminish just in the proportion of the entire load sustained by them. Such 25. a beam will be able to sustain a greater suspended load than any other of the same length and the same area at the top.

For the sections NO, $N_1 O_1$, &c., being every where proportional to the loads, the degree of tension will be the same in all the sections as at $A B$, and the beam will have equal strength throughout. If, now, we suppose it to be widened at $N O$, say to $E F$, this, while it increases the strength of that part of the beam, does not augment that of the whole. For the beam will still break down at $A B$ with the same load as before. But increasing the width of $N O$ we augment the weight of the beam, and, therefore, diminish the load W which it is able to bear. Thus, suppose $A B L M$ and $A B C D$ to be severally loaded so that the tension at $A B$ shall be the same in both cases. Denoting by W and W_1 the weights employed, we have for $A B C D$, the whole load acting on $A B = W_1 +$ weight of $A B C D$; and for $A B L M$, the whole load on $A B = W +$ weight of $A B L M$. Hence $W_1 + A B C D = W + A B L M$, and $W - W_1 = A B C D - A B L M$. That is, the load which $A B L M$ is able to bear exceeds that which $A B C D$ can bear, by the difference of the weights of the beams.

OF THE TRANSVERSE STRENGTH OF BEAMS.

543. In the case of a beam fixed at one end in a horizontal position, and acted on by a weight applied vertically at the other, we have seen that the greatest extension is sustained by the top, and the greatest compression by the bottom of the beam (fig. 276). If $N R$ be the neutral axis, the extension of the fibres or rows of particles between N and A must evidently be greater the farther they are above

N, and the compression of the fibres between *N* and *B* must be greater the farther they are below *N*. With moderate forces we may assume that the extensions in the one case and the compressions in the other are proportional to the distance of the fibres above and below *N*, the neutral axis, and this assumption is approximately true for excessive forces.

Hence Extension at a : Extension at $A = Na : NA$; and so Compression at b : Compression at $B = Nb : NB$. But we have seen (530) that the extending forces are proportional to the extensions, and the compressing forces to the compressions they respectively produce. Hence the Extending force at a : Extending force at $A = Nb : NB$.

Let AK (fig. 277) denote the extending force at A, and draw KN. Then ak will represent the extending force at a; and so the extending force for any other point in NA will be measured by the line drawn from that point parallel to AK in the triangle AKN. The same is true of the compressing forces.

544. To find the position of the resultant of the extending forces, we observe that these forces have the same proportions as the weights of the lines Ka, ka, &c., and hence their resultant will have the same position as the resultant of these weights; it will, therefore, pass through the centre of gravity G of the triangle AKN, and will intersect AN at P, making $NP = \frac{2}{3} NA$. So the resultant of compressing will intersect NB at p, making $Np = \frac{2}{3} NB$.

545. As all the extending forces are parallel and act the same way, their resultant is equal to their sum. So, likewise, the resultant of the compressing forces is equal to the sum of these forces. We may therefore consider the sum of the extending forces applied as a single force at P, and the sum of the compressing forces as a single force at p. Since N is neither extended nor compressed, we may regard it as the fulcrum (fig. 278) around which the two forces above named tend to rotate the beam in one direction, while the weight tends to produce the opposite rotation. By the principle of moments, therefore, we have, Sum of Extending forces $\times NP +$ Sum of Compressing forces $\times Np = LW$. But by (535) Sum of Extending forces = Sum of Compressing forces. Therefore, Sum of Extending forces $\times Pp = LW$.

That is, *the sum of the extending forces multiplied by the distance between the centres of extension and compression, is equal to the length of the beam multiplied by the straining weight.*

546. The sum of the extending forces applied to the different points of the line $A\,N$ is represented by the area of the triangle $A\,N\,K$ (fig. 277). But if the extending force at all points of $A\,N$ were of the same magnitude as at A, that is, equal to $A\,K$, then the sum of these extending forces would be represented by the rectangle $A\,Y$. As the same is true for every other vertical line corresponding to $A\,N$, it follows that the sum of the extending forces, acting on the whole area of extension $A\,N\,A_1 N_1$ (fig. 278), is just one half as great as it would be if the fibres were all in the same state of tension as those at the top.

If then e represent the extending force which acts on a single fibre or unit of the surface at the top of the beam, it is evident that the sum of the extending forces will be equal to one half the product of e by the number of these units in the area of extension. That is,

Sum of Extending forces $= \frac{1}{2}\, e \times$ Area $A\,N\;A_1 N_1$.

Let b denote the breadth, and d the depth of the beam supposed to be of rectangular form. Also let n be the fraction which marks the ratio of $A\,N$ the depth of the neutral axis to the whole depth d. Then we have $A\,N = n\,d$. And the area of Extension $A\,N\,A_1 N_1 = n\,d\,b$. Hence the sum of extending forces is equal to $\frac{1}{2}\,e\,n\,d\,b$. Since $P\,N = \frac{2}{3}\,A\,N$ and $P\,N = \frac{2}{3}\,B\,N$, we have $P\,p = \frac{2}{3}\,(A\,N + B\,N) = \frac{2}{3}\,A\,B = \frac{2}{3}\,d$.

If now these values of $P\,p$, and of the sum of the extending forces be substituted in the above equation (544), we will have

$$L\,W = \tfrac{1}{2}\,e\,n\,d\,b \times \tfrac{2}{3}\,d = \tfrac{1}{3}\,e\,n\,b\,d^2.$$

547. In this expression n is constant for all beams of the same material and of rectangular shape, because, in such case, the depth of the neutral axis bears a constant ratio to the whole depth of the beam. When we desire to compare the weights which different beams are capable of bearing, either for the purpose of measuring their absolute or their

elastic strength, we assume the extending force e, acting on the most extended fibres to be a constant quantity. In the one case, the greatest extending force which the unit will bear without rupture, in the other, the greatest which it will support without being stretched beyond the limits of elasticity. As, therefore, n and e are to be considered constant, $\frac{1}{3}en$ is a constant quantity. For convenience we may use S to denote this constant. We will thus have

$$L\,W = S\,b\,d^2$$

$$\text{and } W = \frac{S\,b\,d^2}{L}.$$

In this expression W stands for the weight or other force, which, applied at the end of the given rectangular beam, produces the degree of extension in the most extended fibre, which corresponds to the value of e.

548. The value of S varies with that of e. When e is the absolute cohesion of the fibre or unit bar of the material, the corresponding value of S is called the coefficient of cohesive strength or tenacity. When e is the elastic strength, the value of S is the coefficient of elastic strength.

Since the strength of the beam is measured by the weight W, which causes a given extension e, it follows that the strength will be proportional to $\frac{S\,b\,d^2}{L}$.

In comparing the strength of rectangular beams of the same material, the coefficient S is evidently the same for all, and hence the strength is proportional to $\frac{b\,d^2}{L}$.

That is, *the transverse strength of a rectangular beam of any given material is proportional to the product of the breadth into the square of the depth, divided by the length of the beam.*

CHAPTER XXVI.

GENERAL VIEW OF THE MECHANICAL CONSTITUTION OF FLUIDS.

COMPARISON OF FLUIDS WITH SOLIDS.

When a force is applied to any part of a solid body, it displaces the molecules on which it acts through a very small space, and in doing so gives rise to a progressively increasing resistance, which balances the applied force, unless the latter be in such excess as to cause a rupture of the mass. The amount of displacement which is capable of destroying the continuity of the solid, and the resistance which the molecules oppose to the change in the different stages of its progress, vary, as we know, with the nature of the solid, and are the cause of the various degrees of toughness, hardness, ductility and elasticity which it presents. In the diamond, this resistance is greater than it is in steel; in steel it is greater than in lead, and in lead than in wax. But in all solids a considerable force is required to displace the particles through a sensible distance.

In soft bodies, such as moist clay or tallow, we find the resistance to displacement much less than in the preceding group, and indeed it is owing to this that they possess their character of softness. In bodies like pitch or tar, we have examples of the still more yielding texture due to a further reduction of the molecular resistance. In these substances we first distinctly mark the tendency *to flow*, or the inability of the mass to retain its form without being supported laterally as well as beneath. Such bodies are recognized as having some degree of fluidity, and are designated by the term *viscous fluids.*

Pursuing this order, we pass insensibly through the numerous gradations of less and less viscous fluids to the group embracing mercury, water, alcohol, ether, &c. in which the resistance, excited by the action of a displacing force, is so small as to be almost imperceptible. Those are the bodies called *liquids* or coherent fluids. In the class of *airs or gases*, including the *vapors* of liquids, such as steam, we find a still greater freedom from restraint in the relative motions of the parts, that is to say, a still more perfect fluidity of the mass.

549. DEFINITION OF FLUIDITY.

We infer, from the above, that a primary characteristic of fluids, as compared with solid bodies, is the slight resistance which they offer to a displacement of their parts. Hence, we may define *a fluid to be a collection of material particles which, though not absolutely capable of free motion among themselves, are so slightly restrained, that the smallest force is sufficient to move them in any direction.*

550. When the force applied to a part of any solid body is such as to cause rupture, or separation to a distance, the unity of the mass is permanently destroyed, and in general it is not practicable, even by again pressing together the separate parts, to cause them to reunite. On the other hand, a part of any mass of fluid, as water or air, may be moved from place to place in the mass without permanent rupture, the space left by it being immediately filled by the flowing in of the surrounding fluid. This tendency to reunite is seen in the readiness with which small portions of water or mercury coalesce or flow together when brought into proximity.

551. THE MOLECULAR TENSION EXCITED IN SOLID BODIES, IS CHIEFLY IN A SINGLE DIRECTION.

We have seen under a previous head (187,) that, when a force *P* is applied to any part *A* of a solid mass, a tension equal to the force is excited in the line of particles, *Aa*, (fig. 279) lying in the direction in which it acts, and that thus the force may be considered as applied in the given direction, at any point in that line. In this sense we may say that a force applied to any point of a solid is transmitted through the mass in its original direction. As the displacement and tension caused in the line of particles *Aa* must bring them into new relations with the contiguous particles, we cannot doubt that the effect of the force *P* will be propagated in some degree throughout the mass, and will give rise to tensions in horizontal and in oblique directions.

In the case of compact and hard bodies like steel, these lateral and oblique pressures are only perceptible when the pressure applied is very great, and then the spreading out of the mass at the sides *B* and *C*, marks the action of a powerful horizontal pressure, resulting from the vertical force. When the solid is *brittle*, it is crushed and ruptured by the applied force, without presenting much lateral expansion. When it is *ductile* and comparatively soft, as in the case of lead or wax, the lateral action becomes more conspicuous, and even a feeble force applied at *A* will be seen to produce a sensible pressure outwards at *B* and *C*. If the lead be contained in a vessel, so as to touch its sides, the lateral pressure, which would otherwise cause it to spread out horizontally, will be seen to act with a *bursting force* against these surfaces. In none of these cases, however, are the lateral and oblique forces, thus excited at the surface of the solid by the pressure *P*, equal to that pressure. It appears, then, that *a force applied in a given direction to a solid mass transmits its action unimpaired only in the line of particles which lie in its direction.*

552. FLUIDS PROPAGATE FORCE EQUALLY IN ALL DIRECTIONS.

If, in the above experiment, we substitute for the solid body a fluid, as water or air, observation shows that the pressure applied to a part of the fluid excites an equal tension every where throughout the mass. This property is commonly expressed by saying, that the *applied force is transmitted equally in all directions.*

To prove this in the case of LIQUIDS, we use a vessel of any form, regular or otherwise, as *MN*, (fig. 280), having a number of openings *A*, *B*, *C*, *D*, *E*, &c. in its sides, to which are attached short tubes of equal sections, with a piston fitting closely, but so as to move with ease. Each of these pistons, except that at *A*, is to be held in its place by a spring capable of being relaxed or tightened by a screw. The vessel being filled with water through the opening *A*, let the several springs be adjusted so as to keep the piston in place, and touching the water. This adjustment serves to balance so much of the force on each piston as is due to the weight of the contained water, increased or diminished, according to position by the weight of a part or the whole of the piston. Replacing the piston *A*, let a pressure be applied on its back, which, together with the weight of the piston, amounts to any given number of pounds P. It will be seen, that all the other pistons are pressed outwards, and that in order to retain them in their places, the springs must be tightened until each presses the piston back with a force equal to P. The same result occurs when the force P is applied directly to B, or any of the other apertures instead of A.

The same truth may be proved in regard to GASEOUS BODIES, by using a slightly modified form of the apparatus above described. The tube attached to the aperture A, where the pressure is to be directly applied, should be of considerable capacity, and the reservoir should be small (fig. 281). Supposing the vessel and tube to be filled with atmospheric air, or any of the gases, let the piston H be pressed downwards in the tube until, by the compression of the air, it exerts any given force P; it will be found that the other equal pistons, B, C, D, &c. previously adjusted in equilibrium, will be pressed outwards with a force equal to P

It thus appears, that *a force applied to any part of the surface of a mass of liquid or gas in an inward direction, excites an equal and outward force at every other part of the surface.*

553. From this we are led at once to a knowledge of the forces excited within the mass. Let a given pressure P be applied at A, (fig. 282), and let $x\,y$ be an imaginary plane within the mass of fluid, and of equal area to A. Draw $x\,m$ and $y\,n$ at right angles to $x\,y$, and conceive $m\,n\,p\,o$ to be a short tube with a piston adapted to it. This piston being equal to $x\,y$, is equal to A, and hence the force P at A (by 552) excites an equal force P_1 on the piston in $m\,p$. The reaction of the piston excites an equal force P_2, which, by the tension of the interposed particles, is transmitted to $x\,y$. As the plane of molecules $x\,y$ is at rest during this action, an equal and opposite force must operate upon it in the direction P_3. The same would be true in every other attitude of $x\,y$; and, therefore, it follows that the pressure applied at A excites equal tensions in all directions throughout the mass.

Thus it results from the observed pressures at the surface, that *whatever be the force applied at one part of a fluid, an* *equal force is transmitted in every direction throughout the mass;* in other words, *a tension is excited in every part of the mass in all directions, and exactly equal to the applied force.*

554. This equal transmission of tension in all directions, is a property of fluids no less essential to their nature than that of the free motion of their parts. It would seem, indeed, to be immediately dependent on the facility with which the particles can be displaced. But until we know more of the molecular structure of fluids than we yet do, we cannot satisfactorily deduce the one property from the other.

555. We may sum up what has been said above, by stating the two characteristic features in the mechanical constitution of fluids to be —

First. — GREAT MOBILITY OF PARTS.

Second. — THE TRANSMISSION OF EQUAL TENSION IN ALL DIRECTIONS.

These properties, demonstrable only by experiment, form the basis of all the laws which govern the action of fluids.

556. Fluids may be divided into three classes: COHERENT FLUIDS, AERIFORM FLUIDS, and ETHEREAL FLUIDS. The existence of the latter class, although it has been rendered highly probable by the explanation it gives of a multitude of facts due to the action of Light, Heat, &c., is still to be regarded as an hypothesis which may or may not be demonstrated by future researches.

557. INQUIRY AS TO THE COHESION OF WHAT ARE CALLED COHERENT FLUIDS. — This class, usually called *Liquids*, comprehends Water, Alcohol, Ether, Oils, Mercury, &c. A mass of such fluid has sensible gravity. The degree of fluidity possessed by different liquids is very unequal. Thus Oils in general are less fluid than Water. Water is inferior to Alcohol, and this is less fluid than Ether. It would seem to be a simple means of comparing liquids in this respect, to mark the velocity with which they severally flow through a small orifice when urged by equal pressures. For other things being the same, it is obvious that the liquid having the most perfect mobility of parts, that is, the greatest fluidity, will flow with the greatest velocity.

But experiments on the flow of different liquids through tubes of very slender bore, have shown that the velocity of discharge under a given pressure is influenced by other conditions beside their relative fluidity. Thus it has been found that through a long capillary tube water flows more rapidly than alcohol, and a dilute solution of sugar in water more rapidly than water, although the alcohol is more fluid than either of the others, and the solution of sugar is the most viscous of the three. This result is to be explained by the different degrees of adhesion of these liquids to the sides of the tube, and the different amount of friction to which they are severally exposed.

558. *The fluidity of a liquid increases as its temperature is elevated.*

This is remarkably seen in the increased discharge of water through narrow pipes, when the liquid has been heated.

559. The term *Coherent* has been applied to this class of fluids, because of the manifestation of an attraction or cohesion among their molecules, and they are usually described as bodies in which the mutual attraction of the particles is slightly greater than their mutual repulsion. When an effort is made to separate the adjacent molecules of a liquid, a slight resistance is encountered, marking the action of an attractive force which tends to restore them to their original distances. But there is no proof here, any more than in the case of solid bodies, that this attraction operates before the space between them has been enlarged. On the other hand, it seems clear that in the quiescent position of the particles, there can be no excess of the attractive over the repulsive force; in other words, no efficient cohesion.

It is certain, that in all liquids the particles are placed at a distance from one another. Hence, if the supposed attraction were to operate, they ought to approach until this attraction ceased, or was balanced by an equal and opposite repulsion. In a word, to maintain the particles at rest among themselves, it is necessary either that no forces act between them, or that equal and opposite forces impel them. It will presently be seen that liquids are capable of being compressed, and that when their molecules are thus made to approach one another a very powerful repulsion is excited between them, increasing rapidly as their distances are lessened. Thus, then, looking only to the mutual forces of the particles, a sound induction would lead us to regard the molecules of liquids as placed at distances forming the limits between attraction and repulsion; so that when drawn farther apart they attract, and when forced nearer together they repel one another.

560. In this view of the relations existing among the molecules of a liquid mass, the effect of their *Weight* has been purposely omitted. But it is easy now to understand what this effect will be.

First. — Let *A B* (fig. 283) denote the surface of any mass of a liquid, as water, contained in a vessel, and *C D E F* horizontal layers at different depths. The liquid in *C D* will be compressed, and that in *E F* more compressed. Hence the contiguous particles throughout the liquid, instead of cohering, will actually be repelling one another with forces depending upon the compression; in other words, upon the depth.

Second. — Let *A B C* (fig. 284) be a small portion of water or other liquid hanging from the surface *A B*. As the whole weight of the drop is suspended from *A B*, it is evident that a force of adhesion must operate between the surface and the layer of liquid next below it, and this force must be equal to the weight of *A B C*. So, likewise, the weight of that part of the liquid which is beneath the horizontal plane *a b* is supported by the attraction operating between this layer of particles and the next below it, and the same of every other layer between *A B* and *C*, the force being of course less as the weight is less; that is, as we approach *C*. These attractive forces between the successive layers are caused by the increased distance to which each is drawn from the one above it, by the weight of the liquid hanging beneath. The tensions at different parts of the drop vary somewhat, as in the case of a solid suspended vertically and stretched by its own weight. It is evident that in the present instance, as in the one cited, the cohesion results from the stretching effect of the weight.

These instances suffice to prove, that *the molecules of a liquid are sometimes mutually repulsive, and at others attrac-*

tive or cohesive, according as gravity or any other extraneous force places them within or beyond the distance of neutrality.

561. MEASURE OF THE COHESION OF LIQUIDS. — Using the term cohesion for the attraction excited between contiguous layers of liquid when their distance is increased, we may render sensible, and may even measure this force, by the following experiment.

From one arm of a balance (fig. 285) suspend in a horizontal position a flat disc of glass, or other substance capable of being wetted by the liquids used, but not chemically affected by them. On bringing it in contact with the surface of the liquid, and then adding weights in the opposite scale until the force thus applied is just enough to separate the disc from the liquid, it will be found that the weight required is sensibly more than is necessary to counterpoise the disc and its adhering film of fluid. This excess is the same with the same liquid, whatever be the nature of the disc, provided the liquid adheres to it. It is, therefore, evidently the measure of the force which is just capable of overcoming the cohesion between the contiguous layers of the liquid which have been pulled asunder.

The following results were obtained by Gay Lussac, the disc used having a diameter of about four inches.

Cohesion of Water 40 grammes, Alcohol 31 gr., Spirits of Turpentine 34 gr.

When the disc and liquid are so related as that the liquid does not attach itself to the surface, then the excess of weight necessary to raise the disc is due to the mutual attraction between the contiguous surfaces of the disc and liquid, and is the measure of this force, which, being entirely different from the attraction of the liquid particles for one another, as measured above, is distinguished by the term *adhesion*.

OF THE COMPRESSIBILITY OF LIQUIDS.

562. In most of the preceding explanations, the compressibility of water and other liquids has been assumed. Indeed, it would seem impossible to form a just mechanical

notion of what takes place in the propagation of tension through a liquid mass, without conceiving it to be the result of a forced approach of the particles to one another; in other words, without admitting the compressibility of the mass. Yet the contrary was for a long time taught, on the authority of a famous experiment of the Florentine Academy.

563. The fact of the compressibility of liquids was first proved by Canton, in the following manner. Using a glass globe having attached to it a long and slender tube (fig. 286), he filled it up to a certain mark on the stem with the liquid he wished to try. On placing it in the receiver of an air-pump and exhausting, he found that the liquid invariably descended to a lower point in the tube, and on restoring the atmospheric pressure, it returned to its previous height. This effect he justly ascribed to the compression of the liquid when under the ordinary pressure of the air, and to its recoil when relieved from that pressure. The pressure applied being small, the whole amount of compression was too inconsiderable to admit of accurate numerical results.

564. Subsequently Mr. Perkins applied to a confined mass of water a pressure amounting to 100 atmospheres. His apparatus consisted of a strong metal tube, closed at one end, and having a rod, passing through a tight collar, at the other. On the rod was fitted a ring, to serve as a register of the compression (fig. 287). The tube being filled, the rod partly immersed, and the cap *A* screwed on, the ring was moved up close to *A*. The pressure was now applied by sinking the instrument to a great depth in the sea. On bringing it up, the ring was found, as in the figure, to have been moved some distance; showing that the water had been compressed to that amount. He also applied the pressure by inclosing the instrument in the interior of a cannon, into which water was driven by a powerful forcing-pump.

565. *Description of Oersted's Piezometer.* — In following up the experiments of Canton, Prof. Oersted contrived an instrument which gives more accurate results than the preceding methods, and which, as used by himself or modified by others, has furnished the numerical measures of the compressibility of liquids at present received.

This instrument (fig. 288) consists of a reservoir *A* of thick glass, and a flask *B* of the same material, with a capillary tube *b* terminating in a small expansion or funnel. This tube is graduated into parts corresponding to a certain fraction of the capacity of the flask, so that a compression equal to one millionth of the whole volume may be read on the attached scale. A manometer *C* is placed beside the tube, to denote by the compression of the air which it contains the amount of pressure which we may be applying. Thus, when the column of air is contracted to $\frac{1}{2}$, $\frac{1}{3}$, &c., we know that the pressure is 2, 3, &c. atmospheres. The mechanism by which the pressure is applied, is a metal piston *m*, moveable by a screw *S* in a cavity in the thick metal cap *D*. A small tube with a stop-cock and funnel *n* is attached to the cap.

To use the instrument, we fill the large tube and the flask with the liquid which is to be compressed. Placing a globule of mercury in the funnel *b* of the capillary tube, we introduce the flask into *A*. We then screw on the cap, allowing the excess of liquid to rise into *r*, and at the same time pumping it back by gradually raising the piston. This adjustment being made, we turn the screw handle *S* so as slowly to force the piston downwards. As the liquid in the outer vessel communicates freely with that in the flask, through the tube *b*, the force applied by the piston excites an equal tension throughout the liquid in both vessels, and, therefore, the liquid in the flask is compressed to the same degree as that in the inclosing reservoir. The progress of this compression is made apparent by the descent of the mercury in the capillary tube, which, like a small piston, interposed between the two masses of liquid, serves to transmit the pressure to the flask, at the same time that it marks the extent to which the volume of liquid in the flask has been compressed. The amount of force brought into action in each stage of the experiment, is determined by the column of air in the manometer.

566. *Of the change in the capacity of the flask, and the correction due to it.*

As the material of the flask itself is compressible, it is necessary to consider the effect which the pressure may have upon its interior volume. To simplify the inquiry, we will suppose the flask to be a hollow cylinder or prism, whose sides are of considerable thickness, and which has a small aperture at the top for communication with the outer liquid (fig. 289). In the first place, by the action of the equal and opposite forces of the external and internal liquid, the sides and bottom and top must be made thinner. Let Δ denote the amount of this thinning or compression. It is evident that the interior diameter of the flask will be increased by this quantity, and so far, therefore, the capacity of the flask ought to be augmented.

But, secondly, it will be seen, that while the outside pressure on ab, pq, de and nm, are respectively balanced by the inside pressures on af, fh, hg and ga, the outside pressures on Am, Bp and Cn Dq, and, again, the outside pressures on Aa, Cc and Bb Dd are opposed to one another. The effect of the former or vertical group is evidently to shorten AB and CD, and, therefore, to bring the top and bottom of the flask nearer together; and the effect of the latter or horizontal group is to shorten AC and BD, and, therefore, to bring the sides nearer together. Their joint effect, therefore, will be to diminish the capacity of the flask. It is easy to see that the reduction of capacity due to this cause greatly exceeds the enlargement due to the thinning of the walls. For the thickness of the walls being in fact a very small fraction of the length AB, or even BD, the shortening of these will vastly exceed the expansion of the interior diameter by the thinning referred to.

Thus, if the breadth BD be 50 times the thickness bf, then BD will be shortened by the amount $50\,\Delta$. That is, the inner diameter of the flask will be lessened by this quantity, while its increase due to thinning will be only d. And if $AB = 500 \times bf$, the vertical sides will be shortened by the amount 500 times Δ, and the internal height be diminished by the same quantity. It follows, therefore, that the joint effect of the inside and outside pressures is to *diminish* the capacity of the flask.

Hence, if the liquid in the Piezometer were incompressible, the index would be forced up in the tube by an amount depending on the contraction of the capacity of the vessel. This contraction disguises a part of the compression in the actual conditions of the experiment, rendering the apparent less than the true compressibility of the liquid. The former is converted into the latter by adding the correction, due as above explained, to the contraction of the flask.

567. The numbers in the Table of Compressibility, which is appended, are the corrected or true compressibility.

TABLE OF THE COMPRESSIBILITY OF LIQUIDS.

NAME OF LIQUID.	COMPRESSIBILITY BY 1 ATMOSPHERE.
Mercury, - - - - -	3.38. MILLIONTHS OF VOLUME.
Sulphuric Acid, - - -	30.35.
Nitric Acid, - - - -	30.55.
Sulphuret of Carbon, -	31.65.
Ammonia, - - - - -	33.05.
Acetic Acid, - - - -	40.55.
Water not freed from Air,	47.85.
Water freed from Air, -	49.65. $= \frac{50}{1000000}$ nearly.
Nitric Ether, - - -	69.85.
Essence of Turpentine,	71.35.
Acetic Ether, - - -	77.65.
Hydrochloric Ether, -	84.25 for 1st atmosphere.
" " -	80.60 for 9th atmosphere.
Alcohol, - - - - -	94.95 for 1st atmosphere.
" - - - - -	91.85 for 9th atmosphere.
" - - - - -	87.35 for 24th atmosphere.
Sulphuric Ether, - -	131.35 for 1st atmosphere.
" " - - -	120.45 for 24th atmosphere.

568. *Diminution of Compressibility with increase of Pressure.*—It will be seen by the table, that when alcohol was subjected successively to increasing pressure, viz. 1. 9 and 24 atmospheres, the 1st atmosphere caused a contraction of 94.95 millionths, the 9th a contraction of 91.85, and the 24th one of 87.35. A similar result occurred in the cases of the other liquids tried in this way, viz. Sulphuric and Hydrochloric Ether. There can be little doubt, therefore, that liquids in general are subject in this respect to the same law as solid bodies, that is, *they become less compressible as they are more compressed.*

569. OF THE ELASTICITY OF LIQUIDS. — Observation shows that whatever be the liquid subjected to pressure in the

Piezometer, it recoils on removing the pressure, and instantly returns to its original volume. We therefore infer, that within the limits of displacement in such experiments, liquids are perfectly elastic bodies. The operation of elasticity in liquids may, however, be more simply shown by the motion of small globules of mercury dropped into an empty glass dish, which are seen to rebound vertically and from the sides of the vessel, like a number of highly elastic spheres of steel or ivory. The same phenomenon occurs with water when the surface of the dish is unctuous, or when coated with any fine light powder. Even rain-drops, under certain circumstances, are found to rebound from the surface on which they impinge. If a large flattened globule of water, resting in a glass cup, previously coated with lamp-black, to prevent adhesion, be agitated, it will assume an oblong form, alternately dilating and contracting along the diameters *A B* and *C D* (fig. 290), so as in these rapid vibrations to have the appearance of the dotted line (fig. 291), and this is still more remarkably shown in water, brought to the spheroidal state. These movements of the globule are like the vibrations of an elastic ball alternately expanding and contracting by the reaction of the particles.

570. But while liquids are thus seen to display perfect elasticity, when confined and pressed upon as in the Piezometer, and when in small masses or globules, where the forces acting do not divide the mass, they show little or no tendency to recoil when a force is applied to any part of a free mass of liquid, as when a rod is struck upon a surface of water. For here, owing to the great facility of motion of the parts, the molecules impinged upon are entirely displaced, before any sensible compression can arise, and therefore before any marked resistance or tendency to recoil can be displayed.

571. Of Aeriform Fluids. — The class of Aeriform fluids includes, along with atmospheric air, the various gases and vapors with which chemistry has made us acquainted. The fluids of this class have sensible gravity, although in a less degree than liquids. The characteristic which distinguishes them from liquid and solid bodies is that their particles are always mutually repulsive, however we may rarefy or con-

dense the mass. Hence a mass of air, if left to its own internal forces, would not, like a liquid or solid under the same conditions, continue of the same bulk and form, but by the repulsion of its parts would rapidly expand in all directions.

572. Aeriform fluids are, moreover, distinguished by being far more compressible than either liquids or solids. Compared with one another they are *equally compressible*, that is, starting with a given pressure common to them all, any equal addition of force will compress them all in the same proportion.

As aeriform fluids are endowed with even greater mobility of parts than belongs to liquids, they evince little or no elasticity when acted on by displacing forces, applied so as to divide the mass. But when they are subjected to pressure in a confined space, they show *perfect elasticity;* expanding as the compressing force is diminished, contracting as it increases, and always returning to the same volume when the pressure is reduced to the same amount.

573. Of Ethereal Fluids. — This class includes heat, light, and the electric and magnetic principles. These agents, if they have an independent existence in nature, or if they be any thing more than mere motions or energies belonging to common matter, must be considered as possessing the essential characteristics of fluids, and in this view they are distinguishable from those of the other classes by their being destitute of gravity, and by their property of permeating dense bodies without deranging the order of aggregation of their particles. The former characteristic has given them the name of Imponderable fluids, or simply Imponderables, which is usually applied in speaking of them as a class.

574. It has been usual among writers on Physics to divide the ponderable fluids into *Elastic* and *Inelastic* or *Compressible* and *Incompressible* fluids. Air and gases are called elastic and compressible, and water and liquids generally are classed as inelastic and incompressible. This distinction originated before the compressibility and elasticity of liquids had been proved, but now that we know them all to be compressible, and that they are perfectly elastic or as nearly so as the gases, this mode of classification can serve only to convey false impressions, and ought to be exploded.

MECHANICS OF LIQUIDS.

575. THE MECHANICS OF LIQUIDS may be divided into HYDROSTATICS, which refers to the pressure and equilibrium of liquids, and HYDRODYNAMICS, which treats of their motion, resistance, impulsive energy, and of the application of this energy in giving motion to machinery. The latter division is more usually called HYDRAULICS.

HYDROSTATICS.

576. The details of this subject may be treated of under the following heads.

I. The Pressure caused by Forces externally applied.
II. The form of a Liquid surface in equilibrium.
III. The Pressure caused by the gravity of the Liquid.
IV. The Equilibrium of Floating Bodies.
V. Application to determining the specific gravities of bodies.

CHAPTER XXVII.

OF THE PRESSURE CAUSED BY FORCES EXTERNALLY APPLIED TO LIQUIDS.

577. MEASUREMENT OF THE AMOUNT OF PRESSURE TRANSMITTED TO A GIVEN SURFACE.

Neglecting any effect which may be due to the weight of the liquid, let us suppose a given pressure p to act on a unit of surface in the side of the vessel $A D$, (fig. 292). By the principle of equal tension (552) it is evident that every other unit of the inclosing surface will be subject to an equal pressure. Let S and S_1 be any unequal areas taken in the inclosing surface, and P and P_1 the entire pressures acting on them respectively. Then, since every unit in S and S_1 sustains a pressure equal to p, we have, $P = Sp$ and $P_1 = S_1 p$. Whence we have, $p = \frac{P}{S}$ and $p = \frac{P_1}{S_1}$, or $P : P_1 = S : S_1$. That is, *the pressures exerted by a liquid at rest, and in a state of tension upon different plane surfaces in contact with it, are proportional to the areas of those surfaces.*

5[illegible]8. This proposition is not restricted to plane surfaces, but is equally true of curved and all other forms of surface. To show this, let AB and CD (fig. 293) be curved or irregular surfaces, forming parts of the inclosure of a confined mass of liquid. We may conceive them to be divided into equal and indefinitely small parts, each of which is a plane. Let a denote the area of each of these parts, n the number of them contained in AB, and n_1 the number contained in CD. Then we have $AB = na$, and $CD = n_1 a$. Whence area AB : area $CD = n : n_1$. By the principle of equal tensions the pressure on all the equal planes is the same. Denoting this elementary pressure by p, we have the whole pressure on $AB = np$, and that on $CD = n_1 p$. Whence we have the proportion, Pressure on AB : Pressure on $CD = n : n_1$. But this we have just seen is the ratio of AB to CD. Therefore, Pressure on AB : Pressure on CD = Area AB : Area CD. That is, *the aggregate pressures exerted on different curved or irregular surfaces by a liquid which acts upon them, in virtue of transmitted tension, are proportional to the areas of those surfaces.*

579. The direction in which the pressure of the liquid acts on each element of the surface, is evidently at right angles to the element; and therefore when the surface is curved, the forces acting at different parts as m and n (fig. 294) are not parallel, as they would be if the surface were plane. Hence, the resultant pressure on AB is less than the sum of the elementary pressure, and therefore less than the resultant pressure on an equal plane surface, CD.

In order to determine the efficiency with which the resultant pressure on a surface, either plane or curved, acts in any given direction, we proceed as follows. Let AB (fig. 294) be any surface, and HP the direction in which the efficient pressure is to be measured. Drawing $A\,a$ and $B\,b$ parallel to HP, and $a\,b$ at right angles to this line, it is plain that, conceiving the liquid to extend to $a\,b$, all the tensions acting at $m\,n$ and other parts of $A\,B$ will, when transmitted to $a\,b$, only press against that surface perpendicularly. Thus $a\,b$ is urged by all that part of the aggregate force on $A\,B$ which is parallel to HP. Hence the efficient force with which the pressure on $C\,D$ operates in the direction HP is equal to the perpendicular pressure on $a\,b$, the projection of the given surface $A\,B$, at right angles to the direction assigned. It follows from this that if parallels $A\,C$ and $B\,D$ be conceived to be drawn intercepting a portion $C\,D$ of the opposite surface of the vessel, the pressures on the two surfaces, $A\,B$ and $C\,D$ counted in the directions $H\,P$ and $K\,P$, will be equal. For the pressures on $A\,B$ and $C\,D$ thus counted, are respectively equal to the pressures on $a\,b$ and $c\,d$, and these last are equal to one another.

It results from this, that the entire pressure acting in a horizontal or other given direction on the surface of a vessel, is equal to the entire pressure acting on its surface in the opposite direction.

580. The tension excited in the liquid and acting on the given surfaces, may be caused either by a force applied to some other part of the inclosing vessel, and thence propagated through the mass, as in the preceding examples, or by a force applied directly to one of these surfaces, which, in this case, assumes the character of a moveable piston, as in (fig. 295). Here the tension acting against the surface S balances the applied force P, and is, therefore, equal to it. But it has been proved that the tensions on S and S_1 are in the same proportion as the areas S and S_1. Therefore the pressure applied to S is to that which it produces on S_1 as S is to S_1. That is, *the pressure applied to a plane surface in any one portion of a liquid, is to that produced by it on a plane in any other portion as the area of the first plane is to the area of the second.*

581. From this it follows, that if S and S be both of them moveable pistons (fig. 296), and any force P be applied to S, the force P with which the other piston will be pressed outwards will bear to P the same proportion as the area S

to S_1. And this is true for every position in which the pistons may be placed. For the tension caused by the pressure acts equally in all directions. Thus in (fig. 297) we still have $P : P_1 = S : S_1$.

To prevent motion on the part of the second piston S_1, we must apply a force P_2 equal and opposite to P_1, the force tending to move it (figs. 296, 297). We thus have equilibrium among the forces applied to the two pistons when $P : P_1$ or $P_2 = S : S_1$. Whence we infer the general proposition, that *two or more pressures, acting by means of pistons or other moveable surfaces upon a confined mass of liquid, will be in equilibrium when they are in the same proportion to one another as the surfaces to which they are applied.*

If the area S, to which we apply a force, be very small compared with the area S from which we are to derive a force, through the propagated tension of the liquid, the force applied will be very small compared with the force produced, and this increase of the latter compared with the former may be carried to any extent by increasing the disproportion between the areas.

582. It thus appears that, in virtue of the equal propagation of the tension through a mass of water or other liquid, we are enabled to multiply a pressure to any required extent, and in effect to produce or overcome the most enormous forces by the application of such as are comparatively insignificant.

OF THE HYDROSTATIC PRESS.

583. One of the most remarkable applications of the principle above mentioned is seen in the engine invented by Bramah, called the Hydrostatic Press. Its principal details are represented in (figs. 298, 299). It consists essentially of a sucking and forcing pump A of small diameter, to the piston of which the pressure is applied, and a large cylinder B with a piston on which the transmitted tension acts, and which, through the plate C, exerts a greatly multiplied force upon the material submitted to its action. The pump and large cylinder communicate by means of the pipe $t\,b$. The pump is worked by a lever L of the second order, whose

fulcrum is at f. The suction-valve at n admits the water which has been cleansed by passing through the strainer m. A valve at r opening upwards, or towards the cylinder, suffers the water to be driven into the cylinder by the pump, but prevents its return. Both the pistons are of solid metal, and move through leather collars, which, by an ingenious construction, are made water-tight.

584. *Action of the Press.* — The lever L being raised, the valve at n opens, and the water ascends, as in the common pump. The lever being then pressed down, the valve at n closes, that at r opens, and the water is forced through the channel $t\,b$ into the large cylinder, where it presses the piston up. The operation is repeated, and thus the piston is caused to ascend as far as may be required. The material to be compressed is placed between the platform C and a cross-piece D, which is fixed in the uprights G and H.

585. *To compute the power of the Hydrostatic Press.* — By what has been proved above, it appears that the pressure applied to the base of the piston of the pump is to that produced at the base of the large piston in the proportion of the area of the former to that of the latter. That is, denoting these pressures and surfaces by P and P_1 and by S and S_1 respectively, we have $P : P_1 = S : S_1$. But the areas of circles being proportional to the squares of their diameters, if we denote by D and D_1 the diameters of the small and large pistons, $S : S_1 = D^2 : D_1^2$.

$$\text{Hence } P : P_1 = D^2 : D_1^2, \text{ and } P_1 = P \times \frac{D_1^2}{D^2}$$

Suppose, for example, that the small piston is ¼ inch in diameter and the large one 12 inches, we shall have

$$P_1 = P \times \left(\frac{12}{\frac{1}{4}}\right)^2 = P \times 144 \times 16 = 2304 \times P.$$

Let us now suppose the force P to be produced on the small piston by the action of a force p applied at the end of the lever L, and let Lf or the power arm be 48 inches, and hf, or the weight arm, be 4 inches, then (by moments) we have $4\,P = 48\,p$ and $P = 12\,p$. This being substituted in the above equation, gives

$$P_1 = 2304 \times 12\,p \text{ or } P_1 = 27648\,p.$$

That is, the force applied at L produces an upward pressure of the platform C, which is 27,648 times greater than that force. Assuming p to be one hundred weight = 112 lbs., the pressure generated at the base of the piston B will be 27648 × 112 pounds, or more than 1382 tons.

By means of an engine such as this, the mere weight of a man's body when leaning on the end of the lever may be made to produce a pressure of upwards of 2000 tons; a force sufficient to lift a large ship with her cargo. The hydrostatic press is the simplest and most manageable of all contrivances for increasing human power, and its force is only limited practically by the inability of the materials to sustain the enormous pressures which it generates.

586. While the principle of the hydrostatic press enables us, by the use of a very feeble force, to balance a very great one, the space through which the less force must be moved, in order to impel the greater one through a given distance, requires to be proportionally increased. Thus, while the piston CD (fig. 297) is moved through the distance Cc, the piston AB must move through a distance Aa, such as to make the volume of water Ab equal to the volume Bd. For as the entire volume of the liquid is not sensibly changed by the pressure, the portion Cd which is forced into the tube at S_1 must be equal to that expelled from the tube at S. But S and S_1 being the areas of the pistons, or of the cross-sections ab and cd of the tubes, the volume of $Ab = Aa \times S$, and the volume of $Cd = Cc \times S_1$.

Hence $Aa \times S = Cc \times S_1$. But it has been proved that $P : P_1 = S : S_1$. Therefore $Aa \times P = Cc \times P_1$, and $P : P_1 = Cc : Aa$. That is, the less force multiplied by the space which it describes is equal to the greater force multiplied by the space which it describes, or, in another form, *the spaces described by the less and greater forces are inversely proportional to the forces.*

If we are to move a load of 10,000 lbs. through any given distance, as D, by the application of a force equal to 1 lb. the piston to which the force 1 is applied must advance through a distance equal to 10,000 D.

587. In the action of forces through the medium of liquids, we have, therefore, the same relation between the forces and

their motions as in the mechanic powers; that is, the power multiplied by the velocity of the power is equal to the weight multiplied by the velocity of the weight. The mass of liquid interposed between the applied force and the pressure which it balances or overcomes, is thus to be regarded in the light of a machine which enables us to establish any desired proportion between these forces, at the same time that the mechanical effect measured by the product of the weight into the space through which it is moved remains the same.

CHAPTER XXVIII.

OF THE FORM OF THE LIQUID SURFACE IN EQUILIBRIUM.

588. Familiar experience shows, that the surface of a liquid which is at rest in a vessel, has the form of a horizontal plane. That this result is a necessary consequence of gravity and the molecular reactions of the liquid, may be thus proved. Let *H R* (fig. 300) be the position of the surface of a mass of liquid. Assuming a particle at *P*, it is evident that by its weight this particle is pressed down and brought within the sphere of repulsion of the immediately contiguous particles. Let the circle *c d*, drawn around *P* as a centre, mark the boundary of this repulsion. Then every particle within this space exerts a repulsive force towards *P*. Drawing *Q P S* perpendicular to the surface at *P*, and taking the particles *a* and *b* in symmetrical positions on opposite sides of this line, it is obvious that the resultant of their repulsions towards *P* will have the direction *P Q*. The same will be true for every other pair of particles situated symmetrically in regard to *Q S*; and since all the particles which repel *P* may be divided into symmetrical pairs, it follows that the entire force of repulsion acting on *P* is in the direction *P Q*. In other words, the molecular repulsion acting on any particle on the surface of the liquid, is at right angles to the surface.

But in addition to the molecular force, each particle is acted upon by its own weight. In order, therefore, that the particle may be held at rest, the molecular repulsion must be directly opposite to gravity. It must, therefore, be in a vertical direction upwards; that is, at right angles to the horizontal plane. But we have just seen that it is also at right angles to the surface of the liquid. It follows, therefore, that the surface of the liquid is parallel to the horizontal plane, or coincides with it.

Hence the surface, instead of resting in the position *HR*, (fig. 300), where the gravity *P G* and the molecular repulsion *P Q* cannot be in equilibrium, must adjust itself to the horizontal plane *HR* (fig. 301), where *P Q* and *P G* are every where opposite and vertical. The condition necessary for the equilibrium of the surface of a liquid is, therefore, simply this, that *the direction of each element of the surface shall be at right angles to the direction of the gravity which acts at the element.*

589. Since the directions of gravity at points far apart on the earth's surface cannot be assumed as parallel, but converge within the earth, the surface of a Lake or Sea is not to be regarded as a plane, but is in reality a curved surface forming part of the general spheroidal surface of the earth. Each particle of water on the lake or sea is urged downwards by a force which is the resultant of its weight and centrifugal force; and thus the equilibrium at the smooth surface of the sea requires that the direction of the surface at each point should be at right angles to the resultant acting at that point.

590. The same condition applies also to a mass of water, which by rapid rotation is made to assume a concave form. Thus, let *A C B* (fig. 302) be the surface of such a whirling mass. A particle *P* on the concave surface is carried round in a horizontal circle, whose radius is *NP*. Let *P Q* measure the centrifugal force caused by this rotation, and *P G* the gravity of the particle, then *P R* will be the direction in which the particle *P* is pressed against the contiguous particles. In order that it may be retained at this point of the curve, the direction of the element of the curve at *P* must be at right angles to *P R* or *P M*. In other words, the surface of the liquid must every where be perpendicular to the resultant force pressing the particles against it.

591. The cavity thus caused by the rotation of the liquid, has the form of a parabola. For, assuming a second point, P_1, and drawing the corresponding lines, and denoting gravity by G, and the centrifugal forces at P and P_1 by F and F_1, we have $G : F = MN : PM$, and $G : F_1 = M_1N_1 : P_1M$, whence $MN = G\frac{PM}{F}$ and $M_1N_1 = G\frac{P_1M_1}{F}$. But the times of revolution of P and P_1 are equal; therefore the centrifugal forces are proportional to the radii; that is, $F : F_1 = PM : P_1M_1$. Hence $\frac{PM}{F} = \frac{P_1M_1}{F_1}$. Therefore $MN = M_1N_1$. Thus the curve formed by the liquid is such that MN, the distance between the horizontal and normal lines, is the same for every point. This is one of the properties of the parabola. It follows, therefore, that the form assumed by a rotating mass of water is that of the parabola. The cavity of an eddy or whirlpool must evidently have the same or a similar configuration.

CHAPTER XXIX.

OF THE PRESSURE CAUSED BY THE GRAVITY OF THE LIQUID.

592. In the preceding inquiries we have purposely omitted to consider the pressures produced in the different parts of a liquid mass by the weight of its molecules. We shall now consider the effect of these pressures alone; that is, we shall investigate the law which governs the distribution of pressure in a mass of water or other liquid, acted on simply by gravity.

OF THE PRESSURE AND TENSION OF A LIQUID AT DIFFERENT DEPTHS.

593. In considering this subject, we will begin by deducing the law of the pressures in different parts of a liquid

mass from the principles already established, and will then present an account of the various experimental proofs.

594. **THE LAW OF PRESSURE DEDUCED FROM THE KNOWN PROPERTIES OF A GRAVITATING LIQUID.** — We assume that all the particles of a liquid gravitate equally, and that the pressure acting on any particle is transmitted by it in all directions around.

Beginning with the simplest case, let us suppose the liquid to be contained in a vessel of upright sides (fig. 303). Let *HR* be its surface, and assume *AB CD*, &c., infinitely thin layers or films of the liquid parallel to *HR*, and, therefore, horizontal. On each small part of *AB*, as *P*, there rests a vertical column of liquid, the whole weight of which presses upon *P*. The vertical pressures on all the equal parts P, P_1 &c., of the layer *AB* are equal. The pressure upon the whole layer is evidently the weight of the prism of liquid extending from *AB* to *HR*, the free surface of the liquid. In like manner we have the pressure upon each element *Q* of the surface *CD* equal to the weight of the column of particles resting upon it; and, therefore, the whole pressure upon *CD* equal to the weight of the volume of liquid *CDHR*. The vertical pressures acting on any equal horizontal surfaces taken in *HF*, are therefore proportional to the depth of these surfaces beneath the free surface of the liquid.

Since all the particles of the liquid are at rest, the downward pressure acting on a horizontal surface within the mass must be balanced by an equal upward one. From the principle of the equal propagation of tension, it follows that the pressure produced at *P*, or any other part of *AB*, will be made to act equally in all directions around it. Thus, the liquid touching the side of the vessel at *A* or *B*, will press horizontally against *A* or *B* with a force equal to that with which it presses downwards. Hence the pressure acting on an element of surface in the vertical side of the vessel, is equal to the weight of a column of liquid having for its base a surface equal to the given element, and for its height, the distance from the element to the free surface of the liquid.

595. In the case just considered, the pressure acting at each element of a horizontal layer, *AB*, of the liquid, may

be directly ascribed to the column of liquid actually extending from the element up to the free surface; and the equal tension throughout the horizontal layer follows from the equality of the elementary columns. Let us now consider the distribution of tension in a horizontal layer of liquid, *when the columns actually resting on its different parts are of different heights.*

In the first place it is obvious, that the pressure on the element *P* (fig. 304), is the weight of the column reaching to the free surface at *M.* Let this pressure be denoted by *p.* In virtue of the principle of propagated tension, the particle at *P* acts upwards and horizontally on the adjoining particles with a force equal to *p.* The upward action of *P* reaches the particle next above, viz. that marked 2 in (fig. 305), undiminished; but when transmitted from the second particle to the third, is diminished by the weight of the second particle. For this weight acts in a direction opposite to *p.* Denoting by *g* the weight of a particle, we have $p - g$ for the force transmitted upwards by 2, and $p - 2g$ for that transmitted by 3, &c.

When, on the other hand, the tension *p* is propagated downwards, it is successively increased by the weight of the particles in the vertical column, thus between the 1st and 2d particle it has the value $p + g$, between the 2d and 3d the value $p + 2g$, &c.

In the horizontal direction, the tension *p* is transmitted without either increase or diminution. For the weight of the particle 2, being sustained by the upward force of the fluid beneath it, can neither aid nor oppose the force with which 1 repels 2, and is in turn repelled by 2. This mutual repulsion in the horizontal direction between 1 and 2 would cause 2 to move towards 3, unless an equal horizontal repulsion operated between these two particles; and the same is true of all the other particles in the same horizontal row or layer. Thus, when there is equilibrium in the fluid mass, a tension equal to *p* must be in action between all the particles in the given horizontal layer *A B.* We may, therefore say, that this tension is propagated in the horizontal direction without change.

596. We see, then, that each particle in the layer *A B,*

(fig. 304), is acted on by the same horizontal tension as the particle *P*. But by the equal propagation of tension these particles act vertically, and in all other directions with the same force as in the horizontal direction. Hence each of these particles acts upon the liquid around it precisely as the particle *P*, which lies vertically beneath the free surface of the liquid. Thus, the particle at P_1 exerts the same pressure around it in all directions as if the column of liquid resting upon it extended to M_1 on the same horizontal plane with *M*; and so of all the other particles in the layer *A B*. The same conclusion is evidently true of any other horizontal layer of liquid in the given mass.

597. The reasoning here used applies to every form of the containing vessel in which a vertical column of the liquid *M P*, however slender, actually connects the upper surface with the horizontal layer beneath. If we imagine this column to include only a single row of particles, it would still produce the same tension at *P*, and therefore the same tension every where in the given horizontal layer as if a column of liquid rested upon this layer, having a base equal to the layer, and extending with a uniform breadth up to the level *M*.

598. By a slight modification in the steps of the proof, it can be shown, that the results above stated are equally true in the case of layers of liquid which, from the peculiar form of the vessel, are connected with the upper surface of the liquid by a column which is not vertical.

Thus in the vessels represented in (figs. 306, 307), the tension or pressure acting on the layer *C D*, is equal to that which would be produced by a column of liquid extending from this surface to *M R*, the level of the liquid in the vessel. For, by the preceding, the pressure on each particle of *A B* is equal to the weight of the elementary column *H B*. Therefore, the particle at *D* sustains a pressure equal to the joint weight of *H B* and *A D*, that is, to the weight of *C M*, and as the same pressure is excited every where throughout the layer *C D*, it follows that *C D* is in the same condition of tension or pressure, as if the column *C D M N* rested upon it. So the pressure on the base *E F* is the same as would be caused by a vertical column having *E F* for its base, and *R K* the vertical height of the liquid for its altitude.

599. The results thus deduced from the mechanical structure of liquids may be summed up in the two following propositions, applicable to every form of the containing vessel.

First. — *When a mass of liquid contained in any vessel is at rest, that is, when the molecular and gravitating forces are in equilibrium, the tension or pressure in any horizontal layer of particles is equal in all parts of the layer.*

Second. — *This tension or pressure is equal to the weight of a vertical column of the liquid, having for its base the area of the given layer, and for its height the vertical distance between the layer and the free surface of the liquid.*

600. EXPERIMENTAL PROOFS OF THE LAWS OF PRESSURE. — The result above stated may be proved experimentally by various simple arrangements, among which the following are best suited for the purpose.

(*a.*) To prove that the tension in any horizontal layer of liquid is equal throughout, and acts equally in all directions.

For this purpose we may use the instrument shown in (fig. 308), consisting of two pieces, a glass tube *A M*, fitted into a brass socket, in which an aperture is ground to admit the end of a bent tube *N*, which at the other extremity expands in the form of a funnel. Over this end *O* is tied a thin film of gum elastic. The tube *N O* is thus capable of being revolved so as to present the surface *O* in a horizontal, oblique, or vertical position, as in *A B C* and *D*. Having filled the instrument up to a point, on the stem, with some liquid lighter than water, as alcohol, it is plunged to any given depth, as *E F* in the position *A*. The column of contained fluid will adjust itself to a certain height *a*, at which its pressure downwards will be balanced by the pressure transmitted through the membrane at *O*. If we now turn *N O* so as successively to give the surface *O* the various positions shown in *B C D*, &c., at the same time keeping the centre of this surface in the same horizontal plane, we will find that the liquid retains the same height *a* above the level of the water, for all parts of the horizontal plane *E F*, and for all directions of the surface *O*; thus proving the pressure to be equal at all parts of the given layer and in all directions. It should be noted, that in the oblique or vertical position of *O*, the pressure on any part above its centre is less, and that on any point below its centre is greater than the pressure at the centre. But as this variation applies equally to the liquid on the other side of the film *O*, it does not affect the inference above deduced.

(*b.*) To prove that the tension or pressure proper to any given horizontal surface is equal to the weight of a vertical column of the liquid, having that surface for its base, and extending thence to the full surface of the liquid.

FIRST FORM OF APPARATUS. — This consists of a brass cylinder *M N*, open at both ends, in which a piston moves freely, so as to be air-tight (fig. 309). This piston is attached by a wire and hook to one arm of a balance, and counterpoised by weights in the opposite scale. A screw in the top of the cylinder enables us to fasten it upon any of the vessels *A B C D*, which, differing in size and shape, have each an open base equal in area to the piston. To use the apparatus we first attach to it the cylindrical vessel *A*, and placing a weight, say one pound, in the scale, we pour water into the cylinder until the weight is counterpoised by the liquid, noting at the same time the height of the column by the graduation on the side of the vessel. It is obvious that, in this case, the pressure on the piston is precisely the weight of the cylinder of water which rests upon it, and this is therefore equal to the counterpoised weight in the other scale. We now succesively attach the other vessels *B C D*, &c., and add water until the same weight is counterpoised, marking in each case the level of the liquid in the vessel. Proceeding in this way, we find that the equilibrium is produced in all when the liquid has been raised to the same vertical height as in the cylindrical vessel. It follows, therefore, that *the same pressure acts on the base of all the vessels, and this pressure is the weight of a cylinder of water resting on this base, and reaching to the free surface of the liquid.*

SECOND FORM OF APPARATUS. — In this elegant and simple arrangement, a syphon-shaped tube, *MP Q* (fig. 310), partly filled with mercury, is used in place of the piston and balance of the preceding apparatus. The short limb of the tube opens into a wide chamber *K*, at the top of which is a screw for attaching the vessels *A B C D*, referred to under the preceding head. A stop-cock *L* serves for emptying the vessel before it is unscrewed. To use this apparatus, we proceed as follows. Having marked the zero point *N*, at which the mercury stands by its own balancing pressure in the two vertical limbs, we attach the cylindrical vessel, and pour water into it until the mercury is driven up in the other limb, say to *n*. Emptying the vessel by *L*, and attaching successively the other vessels *B C D*, &c., we find that on pouring water into them until it reaches the same vertical height as in the cylinder, the mercury is pressed up to the same level *n*. The experiment, therefore, proves that an equal pressure is exerted by the water in all these cases, and that this pressure is equal to the weight of the cylinder of water of the same base and height.

601. RULE FOR COMPUTING THE PRESSURE ON A HORIZONTAL SURFACE.

From what has been proved above, we are at once conducted to the following practical rule for finding the pressure exerted by water against any horizontal surface, either in an upward or downward direction. *Multiply the area of the surface in square feet by its distance in linear feet below the free surface of the liquid;* the result will be the pressure in cubic feet of water, which, multiplied by 62½, the number of pounds in a cubic foot of this liquid, will give the pressure in pounds.

One of the most important, and, at first view, surprising of the results of the principle of pressures which has been proved above, is the fact that the pressure of a liquid against the base or top of the containing vessel is independent of its quantity, and that this pressure may be made to exceed the actual weight of the liquid in any assignable ratio. This result is well illustrated by the experiment called

602. THE HYDROSTATIC PARADOX.

The apparatus used in this experiment is shown in (figs. 311, 312). *A B* and *C D* are broad discs of wood united together by some flexible substance capable at the junctions of resisting a great internal pressure. A vertical tube of small diameter is connected with the chamber *A B C D*, either by a hole in the top, as in (fig. 311), or laterally, as in (fig. 312). Having expelled the air from within by pressing upon *A B*, water is introduced through the vertical tube, and the disc *A B* is pressed up with a force depending on the height of the column in the tube. Suppose the section of the tube to be one square inch, and the area to be 1000 square inches. If, then, the water stands at the height *P*, the upward pressure on *A B* (fig. 312) will be the weight of the volume of water $ABMN = AB \times PQ$. That is, in the present case, this upward pressure will be 1000 times the weight of water *P Q* which produces it. In (fig. 311) since the actual surface pressed upward is 999 square inches, the upward pressure will be 999 times the weight of water in the tube. As in a tube of such dimensions one pound of water would occupy about 16 inches, it follows that a column of water of only 16 inches height and one inch square, would support on the disc *A B* a weight of 1000 pounds.

The pressure thus produced on the upper surface of the vessel is as much greater than the actual weight of the water in the tube producing it, as that surface is greater than the section of the tube.

603. It is easy to demonstrate that although the pressure on the bottom of a vessel may exceed in any given proportion the weight of the contained liquid, the force which is required to support or raise a vessel with its contents, must in all cases be exactly equal to the sum of the weight of the vessel and liquid. Thus, in the case of the liquid *A B C D* (fig. 312), while there is a downward pressure on the base equal to the weight of the volume of liquid *C D M N*, there is an upward pressure on the surface *A B* equal to the weight of the volume *A B M N*. The difference between these pressures is the downward force of the liquid, which we have to overcome in raising the mass. But this difference is obviously the weight of the volume of liquid *A B C D* contained in the vessel. Hence the whole force required for supporting the vessel with its liquid, is the aggregate weight of both.

604. If the vessel be of conical form (fig. 313) and filled with liquid to its apex, the pressure on the base *A B* is equal to the weight of the cylinder of water *A B C D*. But (by

Geometry) the volume of a cylinder is three times that of a cone of the same base and altitude. Therefore the pressure on AB is equal to three times the weight of the liquid filling the cone. But while the base of the cone is pressed down with this force, and would, if moveable, require an equal force applied upwards at S to keep it in place, each point P of the sloping surface of the cone is pressed upwards with a force equal to the weight of a column PM, capable of reaching from that point to CD. The sum of all these upward forces is evidently equal to the sum of all the columns of liquid thus supposed to extend from the sides of the cone up to the horizontal circle CD. That is to say, the whole upward pressure is equal to the weight of the external or supplementary volume of liquid $CAPBD$. The difference between the downward and upward pressures is $ABCD - CAPBD = APB$, the weight of liquid which fills the cone, and this, together with the weight of the vessel itself, is the whole force we are called upon to exert in supporting the cone and contained liquid.

In this case it is evident, that while the downward pressure is equal to three times the weight of the liquid producing it, the upward pressure is equal to twice that weight (fig. 314). If, therefore, the bottom and sides were capable of moving asunder at AB, the former would be pushed down with a force triple, and the latter would be pushed up with a force double the weight of the conical mass of liquid.

605. The great force which an inconsiderable weight of liquid is capable of exerting, is exemplified by the bursting of a stout cask filled with water, and having a long vertical pipe secured into it. On pouring water into the pipe, say to the height P, (fig. 315), an upward force against the head is excited which is equal to the weight of the column $ABMN$, and a still greater pressure is caused against the sides of the cask. If $AB = 2$ square feet and $PQ = 20$ feet, we have $ABCD = 40$ cubic feet of water $= 40 \times 62\frac{1}{2}$ lbs. $= 2500$ pounds for the bursting pressure against the head alone.

606. It is easy to see that this bursting force of water may in certain conditions be brought into operation on an enormous scale, beneath the surface of the earth, especially in

mountainous regions. We have only to imagine a cavity $A B$ of great extent at the depth of two or three hundred feet beneath the surface, or at the base of a hill (fig. 316), and communicating with the higher level by one or more channels $P Q$. If the outlets should become permanently closed, or the supply of water through $P Q$ be so large compared with the expenditure as to fill up the hollow and then the tube to a great height, it is obvious that a pressure would be created against the top and sides of the cavity so enormous as to be capable of forcing up with a kind of explosion the incumbent rock or earth, which would then be in a condition to yield to the transporting violence of the water as it poured out.

607. OF THE COMMON LEVEL OF A LIQUID IN COMMUNICATING VESSELS.

From what has been proved above in regard to the tension of horizontal layers of liquid at different depths, we are readily conducted to another important law in Hydrostatics, viz., that *the free surface of a mass of fluid contained in two or more communicating vessels of any shape or size, is every where in the same horizontal plane; in other words, on the same level.*

The proof of this proposition may be deduced from the foregoing principles in the following manner. Let M and N (fig. 317) be communicating vessels, occupied by the liquid up to the level $H R$ and $H_1 R_1$. Assume any horizontal layer $A B$, and prolong the plane until it meets the perpendicular $H K$ dropped from the level of the liquid in M. By the principle previously demonstrated, an upward pressure is exerted upon $A B$ by the liquid in M, and the communicating tube, which is equal to the weight of a volume of liquid, having $A B$ for its base and $H K$ for its altitude. That is, the upward force is measured by $HK \times AB$. But by the same principle the downward pressure on $A B$ is equal to the weight of a volume of liquid having $A B$ for its base and $H_1 K_1$ for its altitude. This, therefore, is measured by $H_1 K_1 \times AB$. But the upward and downward forces are equal. Therefore $HK \times AB = H_1 K_1 \times AB$. Whence we have HK equal to $H_1 K_1$. But by construction, $B K$ is horizontal. It follows that H and H_1, which are at equal heights above this line, are in one and the same horizontal plane, and, therefore, that the free surfaces HK and HR which (588) are severally horizontal, lie in the same horizontal plane; that is, are on the same level. The same conclusion will follow in whatever part of the liquid we assume the horizontal surface AB, and whatever may be the form of the vessels.

608. A simple and striking mode of demonstrating this truth experimentally, is by the use of the apparatus represented in (fig. 318). A number of glass vessels of various shapes and dimensions, but of equal height, as M N O P, are connected with one another at bottom by means of short tubes which are fastened into a horizontal tube running the whole length of the apparatus. In each of the short tubes is a stop-cock. Suppose all the stop-cocks to be closed, and let water be poured into all the vessels, but to unequal heights. If now any two of them be allowed to communicate by opening the proper stop-cocks, the water will rise in the one and fall in the other until it attains the same level; and if all the communications be opened, the liquid will quickly adjust itself to the same level in all the vessels.

609. The reasoning above used for proving that the free surfaces of a liquid in different communicating vessels are on the same level, serves also to prove that in any one vessel the surface is horizontal. This truth has already been demonstrated by a different procedure. The second mode of proof is as follows. Let HR (fig. 319) be the free surface of a liquid in the vessel HKR. Assume any horizontal layer AB, and conceive it to be divided into equal small elements, ab a_1b_1. It has been proved that the pressure on ab is equal to the volume of liquid $MP \times ab$, and that on a_1b_1 is equal to $M_1P_1 \times a_1b_1$. But it has also been proved that the pressures on these equal parts of the same horizontal layer are equal; therefore $MP \times ab = M_1P_1 \times a_1b_1$. But $ab = a_1b_1$, therefore $MP = M_1P_1$. Hence M and M_1 being at equal distances vertically from the horizontal plane AB, must themselves be in one and the same horizontal plane.

610. The property of liquids now under consideration, is that which is familiarly expressed by saying that *liquids*

seek a level. It is in virtue of this property that water is distributed with such wonderful facility throughout extensive cities, overcoming every obstacle presented by the varying elevation of the ground, rising even to the upper apartments of lofty buildings, and limited only in its ascent by the height of the great reservoir from which its innumerable channels derive their supply. Conveyed in a closed pipe, it may be carried over hills and valleys without the necessity of tunnels to pierce the one, or aqueducts to bridge the other. It is merely requisite that its source shall be higher than any of the eminences over which it is to be carried.

611. It is to the same law of liquid pressure and movement that we are to ascribe the flow of rivers, the motion of the tide as it sends its swell from the deep sea along the shores and into the bays and estuaries of the adjoining land, and the rising and falling of the waves themselves. But there is, perhaps, no more curious agency of this law than that which we see in the supply of water to Springs and Artesian wells.

612. IN ORDINARY SPRINGS the water comes from the adjoining beds of earth, and is derived, as all know, from that portion of the rain which, instead of flowing off at the surface, sinks through the layer of gravel, sand, or other porous materials which lie beneath, until it is arrested by some less pervious bed, usually of clay or of compact rock. Being thus, as it were, dammed up, it fills the layers which lie next the impervious floor, and presses to escape wherever these layers approach the surface. Thus in (fig. 320) the rain falling on the surface *MN*, and filtered through the bed of sand and gravel *MNA B*, is in great part arrested at the upper surface of the clay bed *A B*. The porous layer above this becomes filled like a sponge with the liquid which makes its escape at *A* and *B*, and at other points near the junction of the impervious with the porous stratum. When the floor of the wet stratum has an inclination, the springs are observed to be most numerous and abundant on its lower side, as at *B* in the figure.

613. ARTESIAN WELLS depend for their supplies upon more distant parts of the surface, and are caused by certain peculiarities in the arrangement of the strata. One of these is

the basin-formed stratification (fig. 321), such as exists beneath and around London and Paris. The strata MN and M_1N_1 are composed of *porous* materials, while AB and A_1B_1 consist of such as are comparatively *impervious*. The rain falling at the surface is conveyed through the porous beds, and accumulates in them until the mass of these beds becomes completely charged with water. This water, interposed between the particles of sand and filling numerous minute channels in the strata, exerts at the lower part of the curve a pressure proportional to the depth of this part of the layer beneath the level at which the same layer shows itself at the surface. Thus the rain falling on the area BN is conveyed towards Q, and the whole stratum between ab and cd becomes filled with water. A boring extended to Q in the middle of the basin, will discharge the water in a continued stream at a height above the surface at P, which will depend on the elevation of N or M above this point.

OF LEVELLING, AND THE INSTRUMENTS USED IN THIS OPERATION.

614. In many works of great practical importance it becomes necessary to ascertain and trace a horizontal line, and to compare the level of one point with another. Thus, suppose a spring to be situated at S, and that we wish to know whether the water will spontaneously flow by an aqueduct or by a closed pipe following the surface SLM (fig. 322) to the top of the neighboring hill. Our inquiry can only be answered by ascertaining whether M is above or below the horizontal line SK, which represents the level of the spring. Again, a coal seam is seen in the hill-side at A (fig. 323), and from the general structure of the region is known to be continued at the same height through the opposite hill. We desire, therefore, to find at what point B, a line drawn horizontally from A, would strike the slope LN. Lastly, in determining the relief or topographical features of any region as preparatory to extended lines of improvement, it is necessary to compare the levels of the surface from point to point. The practical method of solving these and similar

questions is called levelling, and the instruments used for this purpose are called Levels. We will first briefly describe the simplest kinds of level, viz., the water level and the spirit level, and then allude to their application to the practical problems above mentioned.

615. OF THE WATER LEVEL. — This, as usually constructed, consists of a bent glass tube *A B C* (fig. 325) mounted on a tripod. When used it is to be partly filled with water, say to *m* and *n*. Since by the hydrostatic principle above proved *m* and *n* will necessarily adjust themselves to the same horizontal plane, it follows that the line *P Q*, found by sighting over the two surfaces, in a tangentical direction, successively to the right and to the left, will be a horizontal line, and will enable us to compare the level of the objects on both sides.

32°

As, in practice, we cannot be sure of great accuracy in sighting along the liquid surfaces *m n*, an addition is made which imparts greater precision to the observations. A float is placed in each of the limbs *A* and *B*, to which is affixed a vertical stem, bearing on the top a sight consisting of two slender wires or threads crossing at right angles. If the points of intersection of the wires be at precisely the same distance above the liquid at *m n*, it is obvious that the right line *x y* passing through these points of intersection must be parallel to *m n*, and, therefore, horizontal. In order to secure this adjustment, we proceed as follows.

Placing the eye behind one of the sights *p*, and directing the view so as to make the cross point of *p* exactly cover that of the other sight *q*, we observe some remote object which is also covered by the point. We then reverse the instrument so as to place *q* next the eye, and we note if the line of view, directed as before through the crossings of the wires, still points to the distant object. Should this be so, the instrument is already adjusted. Should the line of view fall above or below the object, one of the sights must be shifted up or down on the stem, until by repeated observations it is found that the line of sight through the two crossings is precisely the same whichever end of the level may be next the eye. These crossings will then be properly adjusted, and all points seen in the direction of the sight will be in the level of the instrument.

616. Of the Spirit Level. — This is a cylindrical glass tube (fig. 326) closed at the ends and full of alcohol, except a small space which is occupied by air, forming an oblong bubble when the tube is laid on its side. When one end of the tube is higher than the other, the liquid descends and forces the bubble towards the upper end. The tube is not perfectly straight on the top, but rises slightly towards the middle of its length, so that when in a horizontal position the bubble will become stationary at the centre, which is then in a minute degree higher than the rest of the tube. Two parallel lines *a b* and *c d* are engraved upon the tube, at equal distances on each side from the centre and separated by an interval equal to the length of the bubble.

When used in testing the *horizontality of a surface*, it is fastened in a brass or wooden base (fig. 327). When the level is properly adjusted in its base, the bubble will stand between *a b* and *c d*, whenever the base *HR* is placed on a horizontal surface and in whatever direction it may be turned.

When the level is secured in a piece of wood so as to be at right angles to one of the edges of the piece, as *MN* (fig. 328) it forms a very simple *substitute for the plummet*, and is much used for testing the vertical direction of walls, &c. Thus, if *MN* be placed against the surface of a wall, the bubble will assume a central position if the wall be truly vertical, or, if it be not vertical, the bubble will indicate by its motion the direction of its deviation.

The instrument used in measuring the *relative heights of distant points* is a spirit level like that above described, to which is attached a pair of sights, or a small telescope, the whole being mounted on a tripod, (fig. 329). When the instrument has been properly adjusted, the line passing through the points of intersection of the wires in the sights or in the telescope, will be precisely horizontal when the bubble rests at the centre of the tube. This adjustment is effected by directing the view through the crossing and to a remote point, then reversing the end and adjusting and repeating this operation until the line of view is found to be the same whichever end of the instrument is next to the eye.

617. APPLICATION OF THE LEVEL.

First. — Let *S* (fig. 322) denote the position of a spring on an eminence, and suppose that it is required to determine to what height on the neighboring hill-side its water may be conducted by a pipe laid across the intervening hollow. We begin by placing the level near *S*, pointing in a direction towards *M*, and adjusting it so that it is horizontal and on the same level as the surface of the spring. Looking along the sights or through the telescope, we at once discover whether *M* be above or below or on the same level with *S*.

Second. — Let *A* (fig. 323) be the outcrop of a horizontal bed of coal, and suppose that the fallen stones and earth conceal its outcrop on the opposite side, in *B* we plant the level near *A*, and adjust its place so that the base of the coal seam *A* shall be in the line of level. Directing our view through the telescope to the opposite hill, we readily mark a point or points having the same height, and where, therefore, we may confidently look for a continuation of the same coal seam.

Third. — In using the level for determining the relief of the surface, which is its chief employment in the operation of levelling, the instrument is carried successively to different stations, and by a series of short horizontal lines, $m n$, $o p$, &c., (fig. 330) determined by the level, we find the points m n o p q, &c., on the vertical graduated rods or staves. The difference of level between *A* and *H* is, then, computed by adding together the vertical intervals $A m$ $n o$, &c. when they are in the same direction, that is, when they are all upwards or downwards, or by taking their difference when they lie in opposite directions. Thus, $A R = A m + n o + p q$.

618. OF THE DIFFERENCE BETWEEN THE TANGENT LEVEL AND THE HYDROSTATIC OR TRUE LEVEL.

The level determined by means of the instruments just described, is a straight line or plane perpendicular to the direction of the resultant of gravity and the centrifugal force at the point of observation, and is only to be regarded as the hydrostatic level for small distances around the given point. This line or plane represented by $Q P R$ (fig. 331), being tangent to the earth's surface at *P*, recedes continually from the surface when prolonged into space. As the direction of gravity is every where perpendicular to the surface of the earth considered either as a sphere or spheroid, its direction at P will be denoted by $P\,C$ at right angles to the tangent plane, and at Q and R by the lines $Q\,M$ and $R\,N$. A heavy body placed at Q, will, in reality, be on an inclined surface in reference to the vertical $Q\,M$, and will be impelled towards M. The same will be true of a body placed at R. The only point in the tangent level at which the body can remain at rest, is the point P, where the supporting force and gravity are in direct opposition.

Supposing the heavy body to be a mass of water, it is obvious that the liquid would flow from Q and R to P, until its surface $a\,c\,b$ was at every point perpendicular to the resultant of gravity and the centrifugal force. In other words, it would not come to rest until its surface $a\,b\,c$ formed part of the general spheroidal surface of the globe, or was parallel to it. It follows, therefore, that *the hydrostatic level is not a tangent plane, but a curved surface which forms part of, or is concentric with, the surface of the terrestrial spheroid.*

619. To illustrate the difference between the tangent and hydrostatic levels by a practical example, suppose two hills A and B (fig. 332), to be separated by a valley many miles in width, and let us assume that the tangent level taken at the lower summit A will strike the higher summit B, when extended in that direction. If water is to be conveyed from A, across the valley, to the opposite height, it will not rise to the tangent level B, but, as above shown, will adjust itself at the height x or a, so as to bring the two extremities of the column $x\,L\,A$ into the curve $a\,x\,A$, concentric with the earth's surface $P\,M$, and equidistant from it. This may be further proved by assuming a section $r\,s$ of the liquid, at the lowest part of the communicating tube, and comparing the pressure which would act upon it in the direction towards B and towards A, supposing the whole channel $A\,S\,B$ to be filled with water. The pressure towards B is that due to the column $A\,S$, whose height is $A\,P$. The pressure towards A is that due to $B\,S$, whose height is $B\,M$. Thus the latter exceeds the former in the ratio of $B\,M$ to $A\,P$, and the surface at B must subside to the level x or a, so as to make $a\,M = A\,P$. Hence the water

flowing from the reservoir at *A* would be conveyed by the tube to the height *x* or *a*, on the same hydrostatic level, and would fall short of the tangent level *B*.

620. It follows from this, that in order to find the hydrostatic level for points at a considerable distance from one another, it is necessary, after finding the tangent level at one of the stations, to prolong it in the direction of the other, and then to make the proper allowance for the difference between this and the hydrostatic level. It is obvious, that the amount of this difference depends on the arc of the earth's surface, included between the stations. In the distance of one mile the tangent level *A T* (fig. 333) recedes from the surface of the sphere a distance *T B* equal to 8 inches. Hence a canal excavated so that its bottom should coincide with *A T*, would have the water 8 inches deeper at *A* than at *T*, and in order to bring the ends to the same hydrostatic level, the extremity *T* must be cut 8 inches deeper than the tangent plane, that is, must be lowered to *B*. To secure an equal depth of water throughout the whole length of the canal, its bottom should be of a curved form, corresponding with the curvature of the terrestrial spheroid in the region through which it passes. This form, which is in fact the hydrostatic, or true level for the bottom of the canal, is determined with sufficient accuracy for practical purposes, by taking successive tangent levels at small distances apart, and thus forming a polygon of such short sides as to be practically coincident with the curved or true level.

OF THE LATERAL AND OBLIQUE PRESSURE OF A LIQUID ON DIFFERENT PARTS OF A SURFACE.

621. We have already seen that the pressure exerted by a liquid upon any very small part of a surface which it touches, is independent of the altitude of the surface, and is measured by the product of the area of the element into its depth, beneath the free surface of the liquid, and that, moreover, this pressure is exerted at right angles to the element. If, therefore, *A B*, *C D*, *E F*, (figs. 334, 335, 336), be any plane or curved surfaces, exposed to the pressure of a liquid whose free surface is *H H*; and if we assume a_1 a_2 to repre-

sent infinitely small parts or elements of these surfaces, and $d_1 d_2$ to represent the depth of these elements respectively below the free surface HR, the pressure exerted upon them severally will be equal to $a_1 \times d_1$, and $a_2 \times d_2$. Drawing $p_1 p_2$, &c. at right angles to the surface at $a_1 a_2$, &c., and equal in length to $d_1 d_2$, the depth of $a_1 a_2$, beneath the free surface, these perpendiculars will represent the direction and intensity of the pressure acting upon the surfaces at the given points. We are thus conducted to a very simple construction for representing the pressures on different points of a plane surface, and for denoting in certain cases the aggregate pressure on the surface.

622. Let BA be prolonged until it intersects the free surface at M, (fig. 237), and at N, (fig. 338). Draw the vertical AM and in both figures draw AA equal to AM, and at right angles to AB. Join MA, and prolong the line until it meets BB_1, the perpendicular erected at the other end of AB. Then will any line am drawn parallel to AA_1 and BB_1, denote both in direction and quantity the pressure upon the point to which it is drawn. For by similar triangles in (fig. 337) $AA_1 : AM = am : aM$. But $AA_1 = AM$, therefore $am = aM$, the column whose weight measures the pressure upon a. In (fig. 338) we have $AA_1 : am = AN : aN$. But $AN : aN = AM : ah$. Therefore $AA_1 : am = AM : ah$, and since $AA_1 = AM$, we have $am = ah$, the column whose weight measures the pressure on a. In the same manner it may be proved that $BB_1 = BM$, &c., and thus the pressures on the successive points in AB are denoted by the perpendiculars at those points extended to the line $A_1 B_1$.

It is furthermore evident, that the sum of all the pressures upon AB is denoted by the sum of all the columns ab, that is, by the weight of water $AA_1 \; BB_1$ regarded as pressing in a direction perpendicular to the surface.

623. Since the pressure of the water in a vessel ABC (fig. 339) increases as we descend from a to a_1 it is obvious that the vessel should be thicker at a_1 than at a, in order to resist this internal pressure as well at one point as at the other. If it be required to proportion the thickness of the different parts so as to give the vessel an equal

degree of strength throughout, and if we assume that the power of resisting the pressure is exactly proportional to the thickness of the material, we must make the thicknesses ab $a_1 b_1$, &c. proportional to the depths $d d_1$, &c.

624. It is in part for the same reason that a wall or embankment, intended to sustain the pressure of water, is made to expand towards the base. Thus, in (fig. 340), let AH denote the level of the water, and AB the face of the dam, designed to resist its pressure. If we take BM equal to AB, and draw MA, the horizontal lines am $a_1 m_1$ BM, &c. will measure the pressures acting against the points a_1 a, &c. of the dam. Assuming that the strength of the dam at any point would be proportional to its thickness, and supposing EB to be a sufficient thickness to resist the force MB, then by giving the wall the sectional form EBC it will have equal strength throughout. For we have $ab : EB = Aa : AB = am : BM$, and in like manner $a_1 b_1 : EB = a_1 m_1 : BM$. That is, the thickness of the wall at different points is proportional to the pressure acting upon it at these points. In this construction no account is taken of the tendency of the pressures ma and $m_1 a_1$ to overturn the wall. To provide against this action, as well as against inequality in the cohesion of different parts of the wall, it is the practice to make it much wider throughout than ABE, the usual section being of the form $ABCD$.

ON THE AGGREGATE PRESSURE SUSTAINED BY A SURFACE.

625. Every surface, whether plane or curved, may be regarded as made up of an infinite number of infinitely small planes or elementary surfaces. We have already seen that the pressure upon each element is equal to the surface of the element into its depth. Hence the total pressure upon any surface is found by taking the sum of the products of the elementary surfaces into their respective depths. In applying the general rule, we distinguish two cases; first, that of horizontal surfaces; second, that of oblique or vertical planes or curved surfaces.

626. *First.* AGGREGATE PRESSURE ON A HORIZONTAL SURFACE. We have already proved that this is proportional to the

product of the area of the surface into its depth. This measure is obviously equivalent to that above given. For if a_1 a_2, &c. be elements of the horizontal surface *A B* (fig. 341), and *H R* the free surface of the liquid, the aggregate pressure on $A\,B =$ the sum of the pressures on $a_1\,a_2$, &c. $= a_1 \times d_1 + a_2 \times d_2 + a_3 + d_3$, &c. But as $d_1\,d_2$, &c. are all equal to *G H*, the common depths of all parts of the surface, we have $a_1 \times d_1 + a_2 \times d_2 +$, &c. $= (a_1 + a_2, +$ &c.$)$ *G H.* But $a_1 + a_2 + a_3, +$ &c. $=$ whole surface *A B.* Therefore sum of elementary pressures $= A\,B \times G\,H.$

627. *Second.* — AGGREGATE PRESSURE ON VERTICAL OR OBLIQUE PLANES OR ON CURVED SURFACES.

The pressure on any element a_1 is $a_1 \times d_1$, and the entire pressure on the given surface is the sum of all these products; that is, $a_1 \times d_1 + a_2 \times d_2 + a_3 \times d_3$, &c., continued until every element of the surface is included. This is equally true for all attitudes and forms of the surface against which the liquid presses, whether it be the oblique plane *A B C* (fig. 342), or the curved surface *A B C* (fig. 343). To find the value of this sum of elementary pressures, we have recourse to a property of the centre of gravity formerly proved (221).

Let *G* be the place of the centre of gravity of the surface in each of the cases represented in the figures, and let *G H* be the vertical distance of this point from the level of the liquid. Then by the property referred to, *G H* is the average of all the depths $d_1\,d_2$, &c., of the equal elements of the surface below the surface of the liquid. Hence we may assume the surface to be a horizontal plane placed at that depth. From this it follows, by the rule above given (626), that the entire pressure on the surface is measured by the product of *G H* into the area of the given surface.

To give the proof in a more mathematical form, we have by (221),

$$G\,H\,(a_1 + a_2 + a_3, + \&c.,) = a_1 \times d_1 + a_2 \times d_2, + \&c.$$

But $a_1 + a_2 +$ &c. $=$ Area of Surface.

And $a_1 \times d_1 + a_2 \times d_2 +$ &c. $=$ Whole pressure on surface.

Therefore *G H* × *Area of Surface* = *Whole Pressure.*

Hence the following proposition;

The aggregate pressure of a liquid upon any surface whether plane or curved, in any attitude in which the surface may be placed, is measured by the product of the area of the surface into the depth of its centre of gravity below the level of the liquid.

628. To compute in pounds the aggregate pressure of *water* on any immersed surface, we multiply the number of square feet in the area of the surface by the depth of its centre of gravity, counted in linear feet. This gives the pressure estimated in cubic feet of water, and when multiplied by $62\frac{1}{2}$, the number of pounds contained in a cubic foot of water, gives the pressure in pounds.

The following examples, selected for their simplicity and usefulness, will illustrate the application of the principle above proved.

629. First.—*Measure of the pressure on the entire surface of a cube.*

Let AE (fig. 344) be a cubical vessel filled with water. The pressure on the base is equal to the area of the base multiplied by the height; that is, equal to the weight of water contained in the cube. The pressure on any one of the sides, as AM, by the proposition above proved (627), is equal to the area of AM multiplied by GH, the depth of the centre of gravity of this side. But $AM = BE$, the base of the cube, and $GH = \frac{1}{2} AB$. Therefore the pressure exerted on one of the upright sides of the cube is equal to one half the pressure on its bottom; that is, equal to one half the weight of water in the cube. Hence the aggregate pressure on the four upright sides of the cubical vessel is equal to four halves, or twice the weight of water in the cube, and *the whole pressure* on the bottom and sides taken together is equal to *three times the weight of water in the cube.*

630. Second.—*Measure of the Pressure on the entire surface of a vessel having the form of a vertical prism or cylinder.*

As each face of the vertical prism (fig. 345) is a rectangle, we have, Pressure on face = Area of face × GH. Let h denote the height of the prism. Then GH the depth of the centre of gravity is the same for all the faces of the prism, and is equal to $\frac{1}{2}h$. Hence we have Pressure upon all the faces = Aggregate area of faces × $\frac{1}{2}h$ = $\frac{1}{2}$ Aggregate area of faces × h. But Pressure on base = Area of base × h. Therefore whole pressure on bottom and sides of Prism = (Base + $\frac{1}{2}$ Area of Sides) × h.

So in the case of the cylinder (fig. 346), we have the Pressure on the upright surface = Area of upright surface × $\frac{1}{2}h$ = $\frac{1}{2}$ Area of upright surface × h. Pressure on base = Area of base × h. Hence the whole pressure on the bottom and sides of the cylinder = (Base + $\frac{1}{2}$ Area of Upright Surface) h.

631. Third. —*Measure of the Pressure on the sides and base of a Pyramid.*

Let B (fig. 347) be the base of a pyramidal vessel resting on a horizontal surface, and V its vertex, and suppose the vessel to be filled with water up to V. From V draw VD to the middle of AB, and find a point G so situated as to make $GV = \frac{2}{3}DV$. This point is the centre of gravity of the face VAB of the pyramid. Through G draw the vertical HGE from the level HV to the base B. Join DE. The vertical HE is equal to h the height of the pyramid, and since it is perpendicular both to DE and NH, these lines are parallel to one another. Hence $GH : HE = GV : DV$, but $GV = \frac{2}{3}DV$; therefore $GH = \frac{2}{3}HE = \frac{2}{3}h$. The same may be proved of each of the remaining sides of the pyramid. Thus, therefore, the depth of the centre of gravity of any of the sides of the pyramid is two-thirds the altitude of the figure.

But the pressure on the side $ABV = GH$ × area of side = $\frac{2}{3}h$ × area of side; and, therefore, by addition, the entire pressure on all the sides = $\frac{2}{3}h$ × sum of areas of sides = $\frac{2}{3}$ sum of sides × h. The pressure on the base = $B \times h$. Hence we have the entire pressure on the surface of the pyramid = ($\frac{2}{3}$ sum of sides + base) h.

If the form be that of a triangular pyramid, whose four bounding triangles are all equal (fig. 348), we obtain a much simpler expression. In this case the sum of the sides is equal to three times the base, and, therefore, $\frac{2}{3}$ sum of sides = 2 Base; and $\frac{2}{3}$ sum of sides + Base = 3 Base. Hence the entire pressure on the surface of the pyramid = 3 Base × h. But the volume of water which fills the pyramid is (by Geometry) equal to one-third the product of the base and height = $\frac{1}{3}$ base × h. It follows, therefore, that the *entire pressure on the surface of this pyramid is equal to nine times the weight of the water which fills it.*

632. Fourth. — *Measure of the Pressure on the surface of a Sphere.*

If the spherical vessel $A\,B$ (fig. 349) be filled with water, the liquid will exert a pressure against every point of the surface of the sphere, and, therefore, by the proposition above proved, the whole pressure will be measured by the product of this surface into the depth of its centre of gravity from the top of the liquid H. But we have seen under a former head that the centre of gravity of a spherical surface is at the centre of the sphere. Hence $G\,H$ = Radius of the sphere. Therefore we have, *Entire pressure on the surface of sphere = surface × radius of sphere.* But the volume of water which fills the spherical vessel = $\frac{1}{3}$ surface × radius (by Geometry). Therefore *the pressure on the surface of a sphere filled with water is equal to three times the weight of the liquid it contains.*

633. In the preceding examples we have supposed the pressure to arise from the water contained in vessels of cubical, spherical or other form, and, therefore, to be exerted upon the surfaces *from within outwards.* But the result will evidently be the same if the cube or sphere or other form be plunged in water (fig. 350), so as just to immerse the whole surface. The pressure will now be exerted from *without inwards;* but as each point of the given surface is at the same depth below the top of the liquid as in the preceding case, the inward pressure at any point is exactly equal to the outward pressure which it would sustain under the previous conditions. Hence the aggregate pressure on the surface has the same measure as in the preceding examples.

ON THE RESULTANT OF PRESSURES AND THE CENTRE OF PRESSURE OF A PLANE SURFACE.

634. Since the pressure exerted on each element of a sur-

face is at right angles to the element, all the elementary pressures acting on a plane surface must be perpendicular to the plane, and, therefore, parallel to one another. These pressures, therefore, constitute a group of parallel forces all acting the same way. Hence the resultant pressure is in this case equal to the sum of the elementary pressures; that is, to the entire pressure upon the given plane. Moreover, this resultant being parallel to the elementary forces, must have a direction at right angles to the plane. *The point of the plane at which the resultant pressure acts is called the* CENTRE OF PRESSURE.

635. It is a characteristic property of the centre of pressure directly inferrible from the above definition, that if a plane surface pressed upon by a liquid is to be sustained by a single force, this force must be applied at the centre of pressure, in a direction at right angles to the surface, and with an intensity equal to the resultant pressure.

Thus, to retain in its place the loose bottom *B C* (fig. 351) of a vessel full of water, by a prop, we must apply the support at the point *P*, where the resultant of all the pressures acting on *B C* may be regarded as operating. Since in this case the elementary pressures are equal to one another, the centre of pressure is at the middle of *B C*. So, again, if the upright side *A B* be entirely loose, the supporting force must be applied at a point *P*, so situated that the resultant of all the elementary pressures acting on *A B* passes through the point. As in this the elementary pressures are greater the nearer they are to *B*, the centre of pressure, instead of being midway in *A B*, must be lower down than the middle of the surface.

636. It is thus evident that the centre of pressure is not a point fixed in position on the plane, but shifts its place when the direction of the plane is altered. The actual place of the centre of pressure is, therefore, dependent on the *form of the surface* pressed upon by the liquid, and *the position it occupies* in regard to the liquid surface. The determination of its place is often of great importance in hydraulic architecture. In cases like the following, its position is determined by an easy graphical construction.

DETERMINATION OF THE CENTRE OF PRESSURE IN CERTAIN CASES BY A GRAPHICAL METHOD.

637. Beginning with the problem in its simplest form, we will first show how to find the centre of pressure in the case of a right line or very narrow surface placed either vertically or obliquely in the liquid.

638. First. — TO FIND THE CENTRE OF PRESSURE OF A RIGHT LINE, THE UPPER END OF WHICH REACHES TO THE LEVEL OF THE LIQUID.

(1). Let $A B$ (fig. 352) be such a line in a *vertical position.* Draw $B B_1$ perpendicular to $A B$ and equal to it, and join $A B_1$. It has already been shown that $a_1 m_1$ $a_2 m_2$, &c., drawn parallel to $B B_1$ represent the quantity as well as direction of the elementary pressures acting at a_1 a_2, &c. Thus the pressure at a_1 is exactly the same as if the elementary column $a_1 m$ gravitated in the direction $m_1 a_1$ against the point a_1, and so the pressure upon a_2 is equal to the weight of the column $a_2 m_2$ supposed to gravitate in the direction $m_2 a_2$, and so for all the other elementary columns, making the triangular plate of water $A B B_1$. It follows, therefore, that as the pressures upon the various points of the line $A B$ are equal to the weights of the corresponding columns in $A B B_1$ acting horizontally, the resultant of these pressures must coincide in direction with the resultant of the weights of all the columns supposed to act in a horizontal direction. But the resultant of the gravity of the different parts $A B B_1$, in whatever direction we may imagine this gravity to act, must always pass through the centre of gravity of the mass $A B B_1$. Hence the resultant of all the pressures acting on $A B$ must be in a direction passing through the centre of gravity of $A B B_1$. Let G be this point, and draw $G P$ at right angles to $A B$. Since the resultant pressure is perpendicular to the surface at the same time that it passes through G, its direction must be that of the line $G P$, and the point P must be the centre of pressure for the line $A B$. To find the distance of the centre of pressure from A and B, draw $A G C$. Then, since (by Centre of Gravity) $A G = \frac{2}{3} A C$, and since $G P$ is parallel to $C B$, we have $A P = \frac{2}{3} A B$. And, therefore, $P B = \frac{1}{3} A B$. That is, *the* *centre of pressure of a line in a vertical position and extending to the surface of the liquid, is situated at a distance from the lower end of the line equal to one-third its whole length.*

(2). When the line is an *oblique position*, the result is precisely the same. Let $A B$ (fig. 353) be such a line. Drawing $B B_1$ at right angles to $A B$ and equal to $M B$ the depth of the lower end of the given line $A B$, and joining $A_1 B_1$, it is obvious that the pressures on the point a_1 a_2, &c., are measured by the perpendiculars a_1 m_1, &c.; that is, they are equal to the weights of the columns $a_1 m_1$, &c., supposing these weights to act in the directions m_1 a_1, &c. against the corresponding points of $A B$. Hence, as in the preceding case, the resultant of these elementary pressures passes through G, the centre of gravity of the plate of liquid $A B B_1$ in the direction $G P$ at right angles to $A B$. Drawing $A G C$, we have, as before, $A G = \frac{2}{3} A C$, and, therefore, $A P = \frac{2}{3} A B$ and $P B = \frac{1}{3} A B$.

That is, *the centre of pressure of a line extending from the level of the liquid in any oblique direction, is situated at a distance from the lower end of the line equal to one-third its length.*

639. Second. — To FIND THE CENTRE OF PRESSURE OF A RIGHT LINE, WHOSE UPPER END DOES NOT EXTEND AS HIGH AS THE LEVEL OF THE LIQUID.

Let $A B$ (fig. 354) represent such a line or very narrow surface. From A and B draw the verticals $A L$ $B M$ and the lines $A A_1$ $B B_1$ severally equal to these and at right angles to the given line $A B$. Join $A B_1$, and at any point a in $A B$ draw a_1 m_1 perpendicular to this line; we have seen (622) that the pressures at the different points a_1 of a given line are equal to the weights of the columns a_1 m_1, regarded as acting in the direction m_1 a_1, &c. Hence, the whole pressure upon $A B$ is equal to the weight of the plate of liquid $A B B_1 A_1$, and this resultant passes through the centre of gravity of this volume of liquid in a direction at right angles to $A B$. Let the centre of gravity of the figure $A B B_1 A_1$ be found, and suppose it to be at G. Draw $G P$ at right angles to $A B$. This line will mark the direction of the resultant of all the elementary pressures acting on $A B$, and the point P thus found will be the centre of pressure of the line $A B$.

640. OF THE CENTRE OF PRESSURE OF A RECTANGULAR SURFACE.

In treating of this problem, we shall restrict ourselves to those positions of the surface in which its upper edge is on a level with the surface of the liquid or parallel to it.

641. TO FIND THE CENTRE OF PRESSURE OF A RECTANGULAR SURFACE, WHOSE UPPER EDGE IS ON A LEVEL WITH THE SURFACE OF THE LIQUID.

Let $A\ B\ C\ D$ (fig. 355) be a rectangular plane, having its upper edge coincident with the level of the liquid, and let us in the first place suppose it to have a *vertical position.* Draw $B\,B_1$ and $D\,D_1$ at right angles to the plane and equal to its depth $A\,B$ or $C\,D$, and join $A\,B_1$ $C\,D_1$ and B_1D_1. We may conceive the rectangle to be made up of a great number of lines or very narrow surfaces like $r\,s$ parallel to the sides $A\,B$ and $C\,D$. Draw $r\,t$ and $s\,t$ parallel to $C\,D_1$ and $D\,D_1$.

The pressure on any point a of the line $r\,s$ is equal to the weight of the elementary column $a\,m$, and so the whole pressure on this line is equal to all the elementary columns, making up the plate of liquid $r\,s\,t$. As for every other line parallel to $r\,s$ in the rectangular plane, there is an equal corresponding plate of liquid like $r\,s\,t$, it follows that the pressures on the different points of the area $A\ B\ C\ D$ are equal severally to the weight of the elementary columns extending from these points in a direction perpendicular to $A\ B\ C\ D$ and terminating in the surface $A\,B\,C_1D_1$. Therefore, the resultant pressure is equal to the weight of the volume of liquid $A\,B\,C\,D\ B_1D_1$ and this resultant passes through the centre of gravity G, of the given volume, in a direction $G\,P$, which is at right angles to the plane $A\ B\ C\ D$. We have, therefore, only to find the position of the centre of gravity of the volume of liquid here mentioned, and the perpendicular drawn from this point will determine the place P of the centre of pressure.

In (fig. 356) g and g_1 represent the centres of gravity of $A\,B\,B_1$ and $C\,D\,D_1$ respectively. Joining these points, it is obvious that the point G midway in $g\,g_1$ is the centre of gravity of the whole mass of liquid. Drawing $g\,p$ g_1p_1 perpendicular to $A\ B\ C\,D_1$ joining $p\,p_1$, and then drawing $G\,P$ to the middle of this line, the line $G\,P$ will be at right angles to $ABCD$, and will, therefore, be the direction of the resultant pressure. Hence, P will be the centre of pressure. Now, since by (638) $pA = \frac{2}{3}BA$ and $p_1D = \frac{2}{3}CD$, it follows that $PL = \frac{2}{3}LM$.

642. The same result obtains when the rectangle is in *an oblique position* as in (fig. 357). Drawing BB_1 DD_1 at right angles to the rectangle and equal in length to the depths of the points B and D and joining AB_1 CD_1 it is obvious that the whole pressure is the weight of the prism of water $ABDCB_1D$, and this resultant pressure passes through the centre of gravity G of this mass. Drawing GP perpendicular to the rectangle, we find the centre of pressure at P, and, by the same reasoning as above, $PL = \frac{2}{3}Lm$.

643. It follows, therefore, that *the centre of pressure for a rectangular surface whose upper edge coincides with the level of the liquid, has the same position whether the surface be vertical or oblique, and that it is placed in the middle line drawn from the upper to the lower edge at a distance from the former, equal to two-thirds the length of the rectangle.*

644. ILLUSTRATIONS. — In the case of a *Sluice or Flood-gate*, as $ABCD$ (fig. 358), placed either in an erect or oblique position, the centre of pressure P is in the middle line HK, and at a depth PH equal to $\frac{2}{3}$ of HK. A prop or other single force applied at this point will therefore suffice to keep it in place, provided the intensity of the force be equal to the resultant or whole pressure acting upon $ABCD$.

But if the supporting force act at some other point, it must be greater than the resultant pressure in order to sustain the gate. Thus in (fig. 359), let P be the centre of pressure and R the resultant or entire pressure of the liquid, and let a supporting force S be applied at Q. The tendency of the force S is to rotate the gate around H as a centre in the direction SQ, and in order that this may balance the opposite effect of R at P, the moments of the two forces must be equal; that is, $S.HQ = R.HP$ and $S : R = HP : HQ$. That is, the force necessary to sustain the surface in this position exceeds the whole pressure on the gate, in the same proportion as the depth of the centre of pressure exceeds that of the point of application of the given force.

A similar result can be proved for the force S when applied below the centre of pressure. Whence it follows, that in order *to sustain the gate with the least expenditure of power, the force ought to be applied at the centre of pressure.*

645. It results from this, that if the gate be suspended on an axis or pivot $x\,y$ (fig. 358) on a level with the centre of pressure, this axis will be subjected to less strain from the pressure of the water than in any other position. The same considerations show that the beams or braces of such a surface will be most effectual in enabling it to withstand the pressure of the water when, as in (fig. 360), they are placed at equal distances above or below the centre of pressure, and not at equal distances from $A\,C$ and $B\,D$. For the equal resisting force of the beams $m\,n$ and $p\,q$ being applied at equal distances from P, generate a resultant which passes through that point, and thus their joint strength is brought directly to antagonize the resultant pressure of the liquid.

646. TO FIND THE CENTRE OF PRESSURE OF A RECTANGULAR SURFACE, WHOSE UPPER EDGE IS PARALLEL TO, BUT NOT COINCIDENT WITH, THE SURFACE OF THE LIQUID.

When the rectangle does not reach as high as the level of the liquid, but is so placed (fig. 361) as that its upper edge, $A\,C$, is horizontal, and therefore parallel with the surface of the liquid, we find its centre of pressure by the following construction, which is but an extension of the method given in (639) for a right line similarly immersed.

From the angles of the rectangle draw the lines AA_1 BB_1 CC_1 DD_1 perpendicular to its surface, and of lengths equal severally to the vertical distances of $A\,B\,D$ and C from the surface of the liquid. Joining A_1 B D_1 C_1, it is evident that the pressure upon each point of the rectangle is equal to the weight of the elementary column extending perpendicularly from that point to the surface A_1 B_1 D_1 C_1, and that the entire or resultant pressure is the weight of the whole volume of the Trapezoidal prism $A\,D$, regarded as acting every where at right angles to $A\,B\,C\,D$. Hence, as in the previous case, the resultant in question passes through the centre of gravity of this prism of liquid in a direction at right angles to the rectangle $A\,B\,C\,D$, and the point in which a line thus drawn strikes the rectangle is the centre of pressure.

To find the centre of gravity of the prism, first find that of each of the trapezoidal faces. Suppose these to be respectively at g and g_1. Join these points, and it is obvious that G, the middle of this line, is the centre of gravity required. Draw GP perpendicular to the plane AD, and P will be the centre of pressure of this surface. It is evident that, both in this and the preceding case, the centre of pressure P of the rectangle divides the middle line LK in the same ratio in which p the centre of pressure of the line AB divides this line. Hence to find the centre of pressure of the rectangle, the simplest construction is the following. Find the place of the centre of pressure for one of the vertical or sloping sides, as before shown, (639); through this point draw a horizontal line across the rectangle, and find the middle point of this line. This will be the centre of pressure of the rectangular surface.

647. By a similar geometrical construction we may determine the position of the centre of pressure where the surface has the form of *a right-angled triangle.* In treating of this problem we shall limit ourselves to the two simplest cases, viz., when the base of the triangle lies in the surface of the liquid, and when it is parallel to the surface of the liquid and the opposite angle is in that surface.

648. TO FIND THE CENTRE OF PRESSURE WHERE THE SURFACE PRESSED UPON IS A RIGHT-ANGLED TRIANGLE, HAVING ITS BASE IN THE LEVEL OF THE LIQUID.

Let ABC (fig. 361) be the given triangular surface, A the right angle, and AC the base lying in the surface of the liquid. From B draw BB_1 at right angles to the surface ABC, and equal in length to the depth of the point B below the level of the liquid. Join B_1A and B_1C. The weight of the triangular pyramid $ACBB_1$ measures the resultant pressure upon ABC. For draw ac parallel to AC, and aa_1cc_1 parallel to BB_1, and join a_1c_1. Since aa_1cc_1 are parallel to BB_1, they are perpendicular to the plane ABC, and, therefore, to the line ac. Assuming any point m in ac and drawing mn parallel to aa_1, this line will be perpendicular to ABC.—

Since $\frac{a\,a_1}{B\,B_1} = \frac{A\,a}{A\,B} = \frac{C\,c}{C\,B} = \frac{c\,c_1}{B\,B_1}$, it follows that $a\,a_1$ is equal to $c\,c_1$, and that $a\,c_1$ is a rectangle. Hence all lines drawn from points in $a\,c$ and at right angles to the plane $A\,B\,C$, are equal to one another.

Now by construction $B\,B_1$ is equal to the depth of the point B, and since depth of a is to depth of B as $A\,a$ to $A\,B$, it follows that $a\,a_1$ is equal to the depth of a, and so the equal depths of all the points in $a\,c$ are the lines drawn from those points perpendicular to $a\,c$, and forming the rectangle $a\,c_1$. Hence the weight of the rectangular plate $a\,c_1$ of liquid measures the pressure exerted perpendicularly on the element $a\,c$ of this triangular surface. As the pyramid is made up of such elements, the sum of all the pressures against $A\,B\,C$ is equal to the weight of the pyramid. This force, considered as a resultant, must pass through the centre of gravity of the figure.

Let G (fig. 362) be the centre of gravity of the pyramid constructed as above. From B_1 draw through this point the line $B_1 G\,g$ striking the plane $A\,B\,C$ in g. The latter point is the centre of gravity of $A\,B\,C$. Draw $B\,g\,m$, and parallel to $B\,B_1$ draw $G\,P$. Since $B\,B_1$ is by construction at right angles to $A\,B\,C$, $G\,P$ must be so likewise, and therefore P is the centre of pressure of the triangular surface $A\,B\,C$. To find its distance from $A\,B$ and from $A\,C$ respectively, draw $P\,Q$ perpendicular to $A\,B$, and therefore parallel to $A\,C$. Since $g\,G = \frac{1}{4}\,g\,B_1$ we have $g\,P = \frac{1}{4}\,g\,B$. But $g\,B = \frac{2}{3}\,m\,B$. Therefore, $g\,P = \frac{1}{4} \cdot \frac{2}{3}\,m\,B = \frac{1}{6}\,m\,B$. Again, $m\,P = m\,g + g\,P = \frac{1}{3}\,m\,B + \frac{1}{6}\,m\,B = \frac{1}{2}\,m\,B$. But since $P\,Q$ is parallel to $A\,m$, we have $A\,Q : A\,B = m\,P : m\,B$. Hence, $A\,Q = \frac{1}{2}\,A\,B$. So also $Q\,P = \frac{1}{2}\,A\,m = \frac{1}{4}\,A\,C$. *It follows, therefore, that the centre of pressure of a right-angled triangle whose base coincides with the level of the liquid, is at a distance from the base equal to one-half the other leg of the triangle, and at a distance from that side equal to one-fourth the base.*

649. TO FIND THE CENTRE OF PRESSURE WHERE THE SURFACE PRESSED UPON IS A RIGHT-ANGLED TRIANGLE, HAVING ITS BASE PARALLEL TO THE SURFACE OF THE LIQUID AND ITS APEX IN THAT SURFACE.

Let $A B C$ (fig. 363) be the given triangular surface right-angled at A, and let the base $A C$ be parallel to the surface of the liquid and the angle B be in that surface. Draw $A A_1$ and $C C_1$ perpendicular to the plane of $A B C$, and equal respectively to the depths of the points A and C below the surface of the liquid.

Since by hypothesis $A C$ is horizontal, these depths are equal, and therefore $A A_1 = C C_1$. Join $B A_1$ $B C_1$. It is obvious that the whole pressure against $A B C$ is equal to the weight of the volume of water $B A C C_1 A_1$, and that this force, as a resultant, passes through the centre of gravity of the figure. But this figure is evidently a pyramid, having the rectangle $A C C_1 A_1$ for its base and the point B for its apex.

Let g be the centre of gravity of the base. Connect this point with the vertex B by the line $g B$; then will the centre of gravity of the pyramid be situated in this line. Suppose it to be at G. Draw $B m$ $g m$ to the middle of $A C$, and from G draw $G P$ parallel to $g m$. Since g the centre of gravity of the rectangle is equidistant from $A A_1$ and $C C_1$ and m is by construction also equidistant from them, $g m$ is parallel to $A A_1$ and $C C_1$. But these lines by construction are perpendicular to the plane of $A B C$. Therefore, $g m$ is perpendicular to $A B C$, and $G P$ drawn parallel to $g m$ is also perpendicular to $A B C$. Therefore, P is the centre of pressure of the given triangular surface.

To find the distances of P from $A B$ and $A C$ respectively, draw $P Q$ parallel to $A C$, and therefore perpendicular to $B A$. By the parallels $P G$ and $m g$ we have $\frac{B G}{B g} = \frac{B P}{B m}$, and by the parallels $P Q$ and $m A$ we have $\frac{B P}{B m} = \frac{P Q}{m A}$. Hence, $\frac{P Q}{m A} = \frac{B G}{B g}$. But $B G = \frac{3}{4} B g$, therefore, $P Q = \frac{3}{4} m A = \frac{3}{4} \cdot \frac{1}{2} A C = \frac{3}{8} A C$. Again, by the parallels $P Q$ and $m A$ we have $\frac{B Q}{B A} = \frac{B P}{B m} = \frac{B G}{B g}$. But $B G = \frac{3}{4} B g$, therefore $B Q = \frac{3}{4} B A$ and $Q A = \frac{1}{4} B A$.

It follows, therefore, that the centre of pressure of a right-

angled triangle whose apex is at the surface of the liquid, and whose base is parallel with that surface, is situated at a distance from the base equal to one-fourth the other perpendicular side of the triangle, and at a distance from that side equal to three-eighths the length of the base.

ON THE RESOLUTION OF THE PRESSURE INTO HORIZONTAL AND VERTICAL COMPONENTS.

650. The pressure exerted by a liquid upon any right line or plane surface having an oblique position is, as we have seen, at right angles to the surface, and, therefore, oblique in regard to the horizon. The entire pressure considered as a single force may, like any other oblique force, be resolved into a horizontal and a vertical element, the former of which measures the effect of the given pressure in the horizontal, and the latter its effect in the vertical direction. In considering this problem, we will first treat of the Resolution of the Pressure which acts on a right line, and, secondly, of that acting on a plane surface.

651. First. — OF THE HORIZONTAL AND VERTICAL COMPONENTS OF THE PRESSURE ON ANY RIGHT LINE PLACED OBLIQUELY.

Let AB (fig. 364) be a right line, and MN the level of the liquid beneath which it is placed. Let P denote the entire pressure on AB, and H the horizontal, and V the vertical, component. Assume any line pr at right angles to AB, to denote this pressure, and draw pq in the vertical and rq in the horizontal direction. Also draw AC and BC in those directions respectively. Then by the resolution of forces we have this proportion; $H : P = qr : pr$. But by the similar triangles pqr and ABC we have $qr : pr = AC : AB$. Hence, $H : P = AC : AB$. Therefore the horizontal component $H = P \times \frac{AC}{AB}$. Let now G be the centre of gravity of AB. Then by (627) the whole pressure $P = AB \times GH$. Substituting this for P, we have $H = AB \times GH \times \frac{AC}{AB} = GH \times AC$. That is, *the horizontal component of the pressure on a given line is equal to the product* *of the depth of the centre of gravity of the line into the vertical projection of the same.*

652. To find a measure for the vertical component of the pressure P, we have by the resolution of forces, $V : P = pq : pr$. But by similar triangles pqr ABC we have $pq : pr = BC : AB$. Hence $V : P = BC : AB$. Therefore the vertical component of P, viz. $V = P \times \frac{BC}{AB}$. Substituting for P its value $AB \times GH$, we have vertical component or $V = AB \times GH \times \frac{BC}{AB} = GH \times BC$. That is, *the vertical component of the pressure on a given line, is equal to the product of the depth of the centre of gravity into the horizontal projection of the line.*

653. The measure of the horizontal component above deduced is equal to the whole pressure which would be exerted on AC. For let g be the centre of gravity of AC; then whole pressure on $AC = AC \times gh$. But since g and G divide AC and AB similarly, gG is parallel to BC and therefore horizontal. Hence $gh = GH$. Therefore whole pressure on $AC = AC \times GH$, which is the value of the horizontal component. Thus we see that *the horizontal component of the pressure on any line, is equal to the entire pressure which would be exerted by the liquid against the vertical height or projection of the given line.*

654. Second. — OF THE HORIZONTAL AND VERTICAL COMPONENTS OF THE PRESSURE UPON AN OBLIQUE PLANE.

Before we can compute the amount of the pressure as referred to the horizontal and vertical directions, it is necessary to know what ratio the oblique surface bears to its vertical and horizontal projections. This is simply determined as follows:

Let $ABCD$ (fig. 365) be any plane surface, and $abcd$ its projection formed by drawing from the points $ABCD$, &c. of its periphery lines perpendicular to some other plane which may be either vertical or horizontal. Let XY be the line in which the plane of $ABCD$, if extended, would intersect the plane of projection.

Now, imagine two parallel planes, separated by an infinitely small distance to meet XY perpendicularly at r and s, and to intersect the plane of $ABCD$, and its projection in rF, sM and rf, sm. It is obvious that these pairs of lines will be parallel, and that the angle Frf or its equal, Msm, will be the angle of inclination between the plane of $ABCD$ and the plane of projection. It is easy to see that ef is the projection of EF, nm the projection of NM, and the area $efnm$ the projection of $EFMN$.

Since the perpendicular distance between rF and sM and that between rf and sm is every where equal to rs, this line marks the breadth of the elementary space $ENMN$ and that of its projection $efmn$. If, now, we imagine the surface $ABCD$ to be divided by lines parallel to EF into a great number of areas like $EFMN$, each projection will have the same breadth as the area in $ABCD$ to which it belongs, and there will be a projection for every such area.

As rs, the breadth of $EFMN$ and of its projection, is assumed to be infinitely small, we may treat these spaces as rectangles, whose heights are EF ef and whose bases are equal to rs. Hence, $EFMN : efmn = EF : ef$. But since Ee and Ff are both perpendicular to ef, they are parallel to one another. Therefore, $EF : ef = rF : rf$. Hence, $EFMN : efmn = rF : rf$. But it can be shown that every other element of $ABCD$ taken parallel to EF, bears the same proportion to its projection. Hence, the whole surface must be to its projection in this ratio, that is, $ABCD : abcd = rF : rf$. From this we deduce the value of the projection, viz —

$$abcd = ABCD \times \frac{rf}{rF}.$$

Since $\frac{rf}{rF}$ is equal to the cosine of frF, the inclination of the two planes, this expression becomes $abcd = ABCD \times$ cosine of Inclination.

That is, the projection of a given surface on any plane is equal to the product of the area of the surface into the cosine of the angle included between it and the plane of projection.

This proposition is true for every position in which the plane of projection may be placed. But we shall apply it only in two cases, viz., that of the vertical and that of the horizontal direction of the plane.

655. TO FIND THE HORIZONTAL COMPONENT OF THE PRESSURE EXERTED BY A LIQUID AGAINST AN OBLIQUE SURFACE.

Let $A\,B\,C\,D$ (fig. 366) be the given surface, G its centre of gravity at the depth $G\,H$ below the level of the liquid; $S\,Q$ the direction of the horizontal component which is to be computed, and $a\,b\,c\,d$ the vertical projection of the surface on a plane perpendicular to $S\,Q$. Also, let $X\,Y$ be the line of intersection of the planes of $A\,B\,C\,D$ and $a\,b\,c\,d$ supposed to be extended, and let $r\,F$ and $r\,f$ be drawn at right angles to this line.

Assuming $E\,F$ and $e\,f$ to be physical lines, or very narrow surfaces of equal breadth, it is evident that $e\,f$ is the projection of $E\,F$. But (by 651) the whole pressure on $E\,F$ is to its horizontal component as $E\,F$ is to $e\,f$. Again, taking any other parallel element of the surface as $F_1\,F_1$ and its projection $c\,d$, we will have the whole pressure on $F_1\,F_1$ to its horizontal component, as $F_1\,F_1$ to $e\,f$. But the ratio of $E\,F$ to $e\,f$, of $E_1\,F_1$ to $e\,f$, and in general of each element to its projection is that of $r\,F$ to $r\,f$. Hence the whole pressure on each element is to the horizontal component of this pressure as $r\,F$ to $r\,f$. It follows, therefore, that the aggregate pressure on the given oblique surface is to its horizontal component in the same ratio. That is,

Whole Pressure on $A\,B\,C\,D$: Hor. Comp. $= r\,F : r\,f$.

Whence we have, Hor. Comp. $=$ Whole Pressure $\times \frac{r\,f}{r\,F}$.

But (by 627) Whole Pressure $= G\,H \times A\,B\,C\,D$.

Therefore, Hor. Comp. $= G\,H \times A\,B\,C\,D \times \frac{r\,f}{r\,F}$.

Whence by substituting for $A\,B\,C\,D \times \frac{r\,f}{r\,F}$ its value $a\,b\,c\,d$ (654), we have Hor. Comp. $= G\,H \times a\,b\,c\,d$.

That is, *the horizontal component of the pressure exerted by a liquid against an oblique surface, is equal to the product of the vertical projection of the surface into the depth of the centre of gravity.*

656. This result may be presented in another form, thus: By (653) the Horizontal pressure on $E_1\,F_1$ is equal to the whole pressure upon its vertical projection $e_1 f_1$. So likewise the horizontal pressure on $E\,F$ is equal to the pressure on

ef; and as the same is true for all the other parallel elements and their projections, it follows that the horizontal component of the pressure on $ABCD =$ whole pressure on $abcd$. That is, *the horizontal pressure of a liquid against an oblique surface, is equal to the pressure it would exert perpendicularly against the vertical projection of that surface.*

657. TO FIND THE VERTICAL COMPONENT OF THE PRESSURE UPON AN OBLIQUE PLANE.

Let $ABCD$ (fig. 367) be an oblique plane, whose centre of gravity G is at the depth GH beneath the level of the liquid. Let $abcd$ be the horizontal projection of $ABCD$, and XY the line in which the oblique and horizontal planes would intersect if sufficiently extended. Conceive rF and rf to be drawn at right angles to XY. It is obvious that Frf is the inclination of the oblique plane to the plane of projection, and that ef is the projection of EF.

By (652) the whole pressure on EF is to its vertical component as EF to ef, and therefore as rF to rf. As this is true for all the elements of the surface taken parallel to EF, it is true for the entire surface. Hence it follows that the Whole Pressure on $ABCD$ is to its vertical component as rF to rf.

This gives us

$$\text{Vertical Component} = \text{Whole Pressure} \times \frac{rf}{rF}.$$

But (by 627) Whole Pressure $= GH \times ABCD$.

$$\text{Whence, Vertical Component} = GH \times ABCD \times \frac{rf}{rF}.$$

That is, (by 654) Vertical Component $= GH \times abcd$.

Thus the vertical component of the pressure exerted by a liquid upon an oblique surface, is equal to the product of the horizontal projection of the surface into the depth of its centre of gravity.

658. It thus appears, that in computing the force with which water presses in the *horizontal direction* against an oblique surface, we may consider the pressure as applied directly to the *vertical projection* of the surface, and to find the *vertical force*, we may regard the pressure as applied to the *horizontal projection*; but in both cases the head or height

of the column is the distance of the centre of gravity of the given oblique surface from the level of the liquid.

659. MODE OF DETERMINING THE PRESSURE ON A DAM, AND THE STABILITY OF THE STRUCTURE.

The horizontal pressure exerted against the sloping face $BDbd$ of the dam (fig. 368) is equal to the vertical projection $BFbf$ multiplied by the depth of the centre of gravity $= BFbf \,.\, \frac{1}{2}\, BF$. The vertical pressure is equal to the horizontal projection $FDfd$ multiplied by the depth of the centre of gravity $= FDfd \,.\, \frac{1}{2}\, BF$. The dimensions of the dam being given in feet, these measures will be cubic feet of water, which may be converted into pounds by multiplying them each by the number 62.5. Knowing the length and height of the dam and its breadth at base and at top, we compute its entire volume in cubic feet. Let this be denoted by V, and suppose G to be the weight of one cubic foot of the clay or stone of which the structure is composed. Then $V \,.\, G =$ whole weight of dam. Adding to this the vertical pressure of the water on the face $BDbd$, we have $VG + FDfd \,.\, \frac{1}{2}\, BF \,.\, 62\frac{1}{2} =$ entire pressure exerted vertically by the dam upon its base.

Now if C denote the coefficient of friction between the lower surface of the dam and the surface on which it rests, the entire amount of friction at the surface of support, which would oppose the effort of the horizontal pressure, tending to slide the dam from its base, is $= C\,(VG + FDfd \,.\, \frac{1}{2}\, BF \,.\, 62\frac{1}{2})$. In order therefore that the dam may not be pushed away by the horizontal pressure of the water against the face $BDbd$, the friction as just computed must exceed this force $BF\,bf \,.\, \frac{1}{2}\, BF \,.\, 62\frac{1}{2}$.

For safety, it is best to assume the base of the dam as quite permeable to the water, and therefore to deduct from the above estimate of the vertical pressure of the dam upon its base, the upward pressure of the water considered as acting upon the whole base. This would be $CDdc \,.\, BF \,.\, 62\frac{1}{2}$ pounds.

In clay dams it is found a good proportion to make AB AE and EC equal to one another.

EQUALITY OF THE OPPOSITE HORIZONTAL PRESSURES ACTING UPON A BODY PLUNGED IN WATER OR UPON THE SIDES OF A VESSEL CONTAINING WATER.

660. Observation shows that a body suspended in a liquid hangs in the vertical direction, just as if no horizontal pressure were acting upon it. It is therefore obvious, that the pressure in any one horizontal direction must be exactly balanced by that in the opposite direction. So again, when a vessel of any form, containing water, is suspended by a thread, it hangs in the vertical direction, with the centre of gravity of the whole mass under the point of support. Hence the horizontal pressure of the contained water in any one direction, must be balanced by the horizontal pressure in the opposite direction. In either of these cases it is evident, that if the horizontal pressure were not thus balanced, there would be motion in the direction of the greater force.

This equality of the horizontal forces may also be directly deduced, from what has been proved above respecting the measure of the horizontal pressure. In explaining this result, we may consider, first, the horizontal forces acting on a body which is plunged in water, and, second, those which are exerted against the sides of a vessel by the water it contains.

661. First.—*Of the horizontal forces acting on a mass immersed in water.*

Let *R S* (fig. 369) be a mass plunged in water, and therefore pressed upon by the liquid at every point. We may conceive it to be cut up into an infinite number of infinitely slender prisms, *A B C D*, placed horizontally. Whatever be the form of the mass, the surfaces *A B* and *C D* terminating each prism, will be infinitely small planes. Representing them by *M* and *N* in the figure *M P N*, and conceiving the vertical plane *P Q* or *P* to be drawn, this will obviously be the horizontal projection of both the surfaces *A B* and *C D* or *M* and *N*.

By (656) the horizontal component of the pressure on *M* is equal to the pressure which would be exerted on its vertical projection *P*, supposing this to be directly exposed to the liquid. So, likewise, the horizontal pressure on N is equal to the pressure on P, acting on the other side. Hence the horizontal pressure on M is equal to that on N, and as these forces are in opposite directions they must balance one another. The same is true for the pressures on the ends of all the other elementary prisms parallel to $A\,B\,C\,D$, which go to make up the mass $R\,S$.

Hence *the entire horizontal pressure in any one direction is balanced by an equal pressure in the opposite direction.*

662. Second.—*Of the horizontal forces exerted on the sides of a vessel by the liquid contained in it.*

Let $H\,R\,N$ (fig. 370) be a vessel filled with water to the level $H\,R$. Considering the mass of fluid to be divided into small horizontal prisms $A\,B\,C\,D$, we have, as above, horizontal pressure on $A\,B$ = horizontal pressure on $C\,D$; and as these pressures act in opposite directions, they must balance one another. As the same is true for all the corresponding elements of $H\,N$ and $R\,N$, it follows that the whole horizontal pressure exerted by the liquid upon $H\,N$ is equal and opposite to that exerted upon $R\,N$.

663. In the above example, (661,) the immersed body is supposed to be entirely beneath the surface of the liquid. But the equality of the horizontal pressure holds true also in the case of a body which is only partially immersed. Thus in (fig. 371) let $H\,R$ represent the level of the liquid, and therefore $H\,R\,N$ the immersed part of the mass $M\,N$. Dividing this portion of the body into small horizontal prisms, such as $A\,B\,C\,D$, we see that the horizontal pressure on $A\,B$ balances that on $C\,D$, and that thus all the horizontal pressures acting upon $H\,N\,R$, in any one direction, are balanced by those which act in the opposite direction.

664. It follows from this, that any mass floating in water, as, for example, a ship, is pressed equally in the opposite horizontal directions, and is not impelled into motion by the action of these pressures, unless a part of the pressure be withdrawn from one side. This occurs in the case of a leak. Thus, let $H\,N\,R$ (fig. 372) be the immersed part of a vessel, and suppose a portion $A\,B$ in one side to be removed, causing a leak. Let $C\,D$ be the part on the opposite side which corresponds to $A\,B$. It is evident that, in conse-

quence of the removal of $A\,B$, the horizontal pressure on $C\,D$ remains unbalanced. The effect of the leak in this position will therefore be to impel the vessel in the direction $R\,H$, or towards the side of the leak.

665. So again, if a tall tube $H\,R$ (fig. 373) filled with water be suspended vertically, and an opening be made low down on one side, as at $A\,B$, the tube will be deflected from its vertical position towards the opposite side. For in this case the pressure on $C\,D$ is unbalanced, and acts in a direction opposite to that in which the water makes its escape.

OF THE RESULTANT OF THE VERTICAL PRESSURES EXERTED BY A LIQUID ON THE SURFACE OF A BODY WHICH IS PARTLY OR WHOLLY IMMERSED.

666. It is obvious that the resultant must be in a vertical direction, because all the component forces are vertical. It is also evident that it must be equal in amount to the difference between the aggregate of upward and the aggregate of downward forces.

I. Of the amount or value of the Vertical Resultant.

In treating this subject, we will first consider the case of a body which is entirely immersed, and, secondly, that of a body whose surface is only in part exposed to the pressure of the liquid.

667. First. — *To find the resultant of the vertical pressures acting on the surface of a body which is entirely immersed in the liquid.*

In (657) we have shown how to compute the vertical pressure acting on any plane surface, and since the pressure is the same, whether exerted upwards or downwards on the surface, the rule there given applies as well to the former as to the latter direction of the force.

Let $M\,N$ (fig. 374) be a mass of any figure whatever, and $H\,R$ the level of the liquid in which it is immersed. We may consider the whole mass as divided into an infinite number of very slender vertical prisms like $A\,B\,C\,D$, each of which is so narrow that its upper and lower surfaces $A\,B$ and $C\,D$, although parts of the irregular envelope of $M\,N$, are in reality very small planes. Let g and g_1 be the centres of gravity of these planes, join $g\,g_1$ and draw $g\,h$

perpendicular to HR. Also draw PQ to represent a horizontal section of the prism, and draw ab and cd parallel to it. They will each obviously be equal to PQ.

By (657) the vertical pressure on AP is equal to $PQ \times gh$, or which is the same thing to $ab \times gh$. That is, this vertical pressure is equal to the weight of a column of liquid having ab for its base and gh for its altitude. In like manner the vertical pressure exerted on CD is equal to $cd \times g_1 h$, that is, to the weight of a column of liquid having cd for its base and $g_1 h$ for its altitude. As the former force is exerted downwards, and the latter upwards, their combined effect will evidently be measured by their difference, and will be directed vertically upwards. This difference $gh \times cd - g_1 h \times ab$ is equal to $gg_1 \times ab$, that is, to the weight of a column of liquid having cd for its base and ac for its altitude. But when AB and CD are infinitely small, $ABCD$ is equal to $abcd$. Thus, therefore, the effect of the opposite vertical pressures on AB and CD is to produce an upward force equal in amount to the weight of the elementary prism $ABCD$, supposing it to consist of the liquid.

In the same manner it may be shown that the opposing vertical pressures at the end of any other elementary prism $A_1 B_1 C_1 D_1$ give rise to an upward force measured by the weight of this volume of the liquid. It follows, therefore, that the *Resultant* of all the vertical pressures acting on MN, that is to say, the aggregate excess of the upward over the downward pressures of the liquid on the different parts of the mass, is equal to the sum of the weights of the elementary prisms of liquid $ABCD$ $A_1 B_1 C_1 D_1$, &c.; in other words, to the weight of a volume of liquid which would fill the space occupied by MN. It is obvious that this resultant is directed vertically upwards.

What has here been proved is equivalent to the following proposition, viz: *When a body is wholly immersed in a liquid, the sum of the upward exceeds the sum of the downward pressures exerted upon its surface, by a quantity equal to the weight of the displaced volume of liquid.*

668. From what has been just proved, it follows that when a body is entirely immersed in a liquid, we may regard the

opposing forces on the upper and lower parts of the surface as withdrawn, provided we conceive each element of its lower surface, as CD, to be pressed upwards with a force equal to the weight of the corresponding volume of liquid $ABCD$. So, likewise, *we may regard the whole mass as being pressed upwards with a force equal to the weight of a volume of liquid, having the same dimensions as the mass itself.*

669. This principle, which is said to have been first demonstrated and practically applied by Archimedes, is usually expressed by saying that *a body, when immersed in a liquid, sustains a loss of weight equal to the weight of liquid which it displaces.* It is quite evident that in reality the weight of the body is unimpaired by the immersion, but so much of that weight, as is equal to the weight of the same bulk of liquid, is supported by the liquid, because, as we have just seen, the upward pressure on the body exceeds the downward by this amount.

670. Experimental proof of the principle of Archimedes.

The truth of this principle is easily demonstrated by the following experiment. Having provided two similarly graduated vessels R and S, place one of them, R, on the scale-pan of a delicate balance, and fill the other with water to a given level, AB (fig. 375). Let a mass M of any solid heavier than water, and insoluble in it, be suspended by a fine thread or silk fibre, beneath the same scale-pan by means of a hook, and in this position let it be accurately counterpoised. Sliding the vessel S to a suitable place below the scale, allow the mass M to be immersed in the water contained in it. The liquid will be driven to some higher level as ab, and as the surface previously stood at AB it is obvious that $ABab$ is the volume of water which has been displaced by M, and therefore is equal to the bulk of this solid. While M is undergoing immersion, it will be noticed that the end of the balance from which it hangs is caused to rise, and this effect, which is obviously due to the supporting force of the liquid, augments until the mass is entirely submerged. Let water be now poured into the upper vessel until it restores the equilibrium. It is evident that the weight of the liquid thus added is precisely equal to the supporting force of the water, that is, to the apparent loss of weight sustained by the solid. On marking the volume $CDcd$ of the water which has been poured into R to restore the equilibrium, it is seen to be precisely equal to $ABab$, that is, equal to the volume of the immersed mass **M.** Therefore the body M has sustained by the immersion an apparent loss of weight equal to the weight of the same volume of water.

671. Second. — *To find the resultant of the vertical pressures acting upon the surface of a body which is only in part immersed.*

Let MN be the body plunged in the liquid to the level HR (fig. 376), when, as in the figure, the breadth of the body is greatest below HR, draw the verticals HK and RS. The whole immersed volume is thus divided into HMK and RNS, on which the liquid presses downwards, as well as upwards, and $HKSR$, upon which it presses upwards only. By the preceding principle the excess of the upward over the downward pressures acting on HMK is equal to the weight of the liquid whose volume is HMK; and the excess of the upward over the downward pressures acting on RNS is equal to the weight of liquid whose volume is RNS.

It remains now to determine the vertical pressure on the surface KS, which is evidently in an upward direction only. We may conceive $HKSR$ to be divided into an infinite number of very slender vertical prisms like $ABPQ$. The lower surface, AB, of each of these may be regarded as a plane. Let g be the centre of gravity of AB, and draw gh perpendicular to the surface of the liquid. By (657) the vertical pressure on AB is equal to $PQ \times gh$, that is, to the weight of a prism of liquid $ab\ PQ$, which has PQ for its base and gh for its altitude. But when AB is infinitely small, $ab\ PQ$ is equal to $ABPQ$. Therefore the vertical pressure on AB is equal to the weight of a prism of liquid whose volume is $ABPQ$.

As the same is true for all the other elementary surfaces in KS, it follows that the whole upward pressure on KS is equal to the weight of a mass of liquid whose volume is

HKSR. But we have just seen that the excess of the upward over the downward forces, acting on *HMK* and *RNS*, is equal to the weight of a mass of liquid having the volume of these two spaces united. Therefore, *the resultant upward pressure, or the excess of the whole upward over the whole downward pressure of the liquid, on the immersed body, is equal to the weight of a mass of liquid of the same volume as the immersed part of the body.*

672. When the mass is of a form which does not grow broader below the level of the liquid, (fig. 377,) there are no parts corresponding to those marked *HMK* and *RNS* in the preceding figure. The pressures on all the elements of the immersed surface are vertically upwards, and the vertical pressure on each, as for example that on *A B*, is equal to the weight of a prism of liquid *A B P Q* corresponding to the given element. Hence, *the resultant vertical pressure is equal to the weight of a mass of liquid whose bulk is the same as that of the immersed part of the body.*

673. EXPERIMENTAL PROOF OF THE MAGNITUDE OF THE VERTICAL RESULTANT, IN THE CASE OF A BODY PARTLY IMMERSED.

To prove the above proposition experimentally, we make use of an arrangement (fig. 378) like that employed for demonstrating the principle of Archimedes in the case of bodies wholly immersed. But in order to adjust the immersion of the mass *M* to any particular point, a stiff wire is passed through the cord in a horizontal direction, and in such a position that when the mass has sunk to the intended depth, the wire, by resting on the edge of the vessel, will prevent any further descent.

Having placed the empty vessel *R* on the scale-pan and suspended the body by the hook below, we add weights on the opposite side until there is equilibrium. We then bring the vessel *S*, which is charged with water up to *A B*, into a position beneath the body *M*, and raise it so as to cause the partial immersion of the mass. Lastly, we pour water into the upper vessel *R* until the horizontal wire is brought very nearly in contact with the edge of the lower vessel. On marking the rise of level in *S*, caused by the immersion of *M Q* a part of the mass *M*, we find that the volume *A B a b* is exactly equal to the volume *C D c d* of the water which has been poured into the upper vessel. But it is evident that $A B a b$ is equal to $M Q$, the immersed part of the solid. Hence the volume $C D c d$ is equal to the immersed part of the solid, that is, to the volume of liquid displaced by the immersion. But, by the experiment, we see that the weight of the water $C D c d$ is necessary to counterpoise the upward action of the liquid on the immersed part of the solid. These two forces must therefore be equal. Whence it follows that the body is pressed upwards with a force equal to the weight of the water which it displaces.

674. It thus appears that the principle of Archimedes applies to all cases, and that *whether a body be partly or wholly immersed in a liquid, it sustains an effective upward force, or suffers an apparent loss of weight which is equal to the weight of the displaced volume of liquid.*

675. OF THE RESULTANT OF THE VERTICAL PRESSURES EXERTED BY A LIQUID ON THE SIDES OF THE VESSEL CONTAINING IT.

In this case the pressures act from within outwards, and not from without inwards, as in the case of an immersed body. The downward pressure is exerted on the lower part of the interior surface, and the upward upon the upper part of this surface. The former being of course the greater of the two, the resultant vertical pressure is the excess of the downward over the upward pressures.

Let $M N$ (fig. 379) be a vessel of any form filled with water as high as V, or the level $H R$. Assuming $A B$ and $C D$ to be elements of the surface included between the parallel vertical planes $A C$ and $B D$, we have by (657) the upward pressure on $A B = g h \times P Q$, and the downward pressure on $C D = g_1 h \times P Q$.

The difference of these is $g g_1 \times P Q$, which is equal to $A B D C$. Hence as regards the elements $A B$ and $C D$, the resultant vertical pressure is equal to the weight of the volume of water $A B D C$. As the same may be proved for all the other elements of the upper and lower parts of the surface, it follows that the sum of the downward pressures exceeds the sum of the upward ones by a quantity equal to the weight of the water contained in the vessel. In other words, the resultant of the vertical pressures of a liquid on the interior of a vessel which it fills, is equal to the weight of the liquid.

If the vessel be only in part filled, as in (fig. 380) we may conceive the verticals HK and RS to be drawn to the surface of the liquid. Then, by the above proposition, the vertical resultant for the surface HMK is equal to the weight of the volume of liquid HMK, and that for the surface RNS is equal to the weight of the volume of liquid RNS. The vertical resultant for the remaining surface KS is evidently the sum of the weights of all the elementary prisms $ABPQ$ resting on that surface, and is therefore equal to the weight of the volume $HRSK$. Hence, the whole vertical resultant is equal to the entire weight of the contained liquid.

II. OF THE POSITION OF THE RESULTANT OF VERTICAL PRESSURES.

In the preceding paragraphs we have determined the amount or magnitude of the vertical resultant, and have reduced all the cases of immersion to the same general principle. It remains to ascertain the place of this resultant in relation to the immersed part of the body.

676. First. — *When the body is wholly immersed.*

Let MN (fig. 381) be such a body, and conceive it to be divided into an infinite number of vertical prisms, as $ABCD$ and $A_1B_1C_1D_1$. We have seen, under a former head, that the vertical resultant of the pressures on AB and CD is equal to the weight of the volume of water $ABCD$, and is directed upwards. So the vertical resultant for A_1B_1 and C_1D_1, is equal to the weight of the volume $A_1B_1C_1D_1$, and is directed upwards; and the same is true for all the other elements of the immersed surface. Let R denote the resultant of all these parallel upward forces, and suppose G to be the place of the centre of gravity of the volume MN.

We know that the *resultant of the weights* of $ABCD$, and the other elementary prisms, passes through the centre of gravity G in a direction vertically downwards. But we have just seen that the elementary upward forces are equal to these weights respectively, and differ from them only in acting the opposite way. It follows, therefore, that their resultant R passes through G in a direction vertically upwards.

677. Second. — *When the body is partly immersed.*

Let MN (fig. 382) be immersed to the level HR. The vertical pressure of the liquid upon each element AB of the immersed surface, is equal to the weight of the prism of liquid $ABPQ$, and is directed vertically upwards. The resultant of all these vertical pressures must, therefore, have the same position as the resultant of the weights of all the elementary prisms, but with an upward instead of a downward direction. It must, therefore, pass through G, the centre of gravity of the immersed volume.

678. It thus appears, that *whether a body be wholly or partly immersed in a liquid, the resultant of the vertical pressures exerted against its surface passes through the centre of gravity of the immersed volume, and is directed upwards.*

CHAPTER XXX.

OF THE EQUILIBRIUM OF FLOATING BODIES.

679. We have seen in the preceding chapter, that when a body is wholly or partly immersed in a liquid, all the vertical pressures of the liquid upon the immersed surface form a single vertical resultant; that this resultant is directed upward through the centre of gravity of the volume immersed; and that it is equal to the weight of this volume of the liquid.

It is evident that this vertical resultant is equivalent to the entire sustaining force exerted by the liquid upon the body. It is hence called the *Force of Buoyancy*, or the buoyant force of the liquid, and the point through which it acts, that is, the centre of gravity of the immersed volume, is called the *Centre of Buoyancy.*

680. In order that a body, which is wholly or partly immersed in a liquid, may remain at rest, two conditions are necessary.

First. — *The Force of Buoyancy must be equal to the weight of the Body.*

Second. — *These two forces must act in the same vertical line;* in other words, the centre of Buoyancy and the centre of Gravity of the body must be situated in the same vertical line.

These conditions result immediately from the principle, that when two forces balance one another, they are at the same time equal and directly opposite to one another.

681. *First condition.* — Let *M N* (fig. 383) represent a body partly immersed, *G* its centre of gravity, and *B* the centre of gravity of the immersed volume, that is, the centre of buoyancy for the given position of the mass. It is obvious that the force of buoyancy acts vertically upwards through *B*, while the resultant gravity of the floating mass acts vertically downwards through *G*. When the weight is greater than the buoyant force, the mass sinks deeper in the liquid, when it is less, the mass ascends; and in either case it finally adjusts itself to such a degree of immersion, as to make the force of buoyancy and the weight of the body precisely equal.

Like considerations apply when the body is entirely immersed. Let *M N* (fig. 384) be a mass in this position, and let *B* and *G* be the centre of buoyancy and the centre of gravity of the mass. If the force of buoyancy acting upward through *B* should be equal to the force of gravity acting downward through *G*, it is evident that the body will have no tendency either to rise or to descend in the liquid. If, however, the force of buoyancy be greater than the weight, the body will continue to ascend until it in part emerges from the surface of the liquid, and by this reduces the buoyant force to equality with the weight. If the weight be greater than the force of buoyancy the body must descend.

682. *Second condition.* — When, as in (fig. 385), the points *B* and *G* are not in the same vertical line, the force of buoyancy represented by *B P* and the weight represented by *G W*, although they may be equal, cannot be in equilibrium. For it is evident that they both tend to rotate the mass in the direction denoted by the lines *G g* and *B b*. They must therefore continue to change the attitude of the body until the points *B G* are brought into the same vertical

line, and the weight and buoyant force are finally in equilibrium.

683. We have seen (674), that the effective supporting force of a liquid, in other words, its force of buoyancy, is in all cases equal to the weight of the liquid which is displaced. But it has just been shown that the weight of a floating mass and the force of buoyancy of the surrounding liquid are equal. Hence the weight of the mass and that of the displaced liquid are equal. Thus, instead of the first condition of equilibrium above stated, we may substitute the following proposition, viz.:

When a body floats in a liquid so as to be either partly or wholly immersed, the weight of the displaced liquid is equal to the entire weight of the floating body.

This important proposition is not only deducible from the laws of liquid pressure as shown above, but may be proved experimentally by the following very simple arrangement.

EXPERIMENTAL PROOF.

Let a cylindrical vessel (fig. 386), containing water up to the level *A B*, be counterpoised in one of the scales of a balance, by weights added on the other side. A mass of wood or other body *M N* capable of floating in the water, being now introduced into the vessel, let weights be added in the opposite scale until the equilibrium is restored. Suppose *Q N* to be the part of the floating body which is immersed, and *a b* the level to which the surface of the liquid is raised by the immersion of *Q N*. Having made a mark at *a b* we withdraw the solid, and by so doing cause the other scale to preponderate. If now water be added until its surface is again raised to *a b*, we find that the equilibrium is reinstated. It is evident from the experiment that the weight in the scale *W* is equal to the weight of the water *A V B* plus that of the water *a b A B*, and that it is also equal to the weight of the water *A V B* plus that of the body *M N*. It follows therefore that weight of the body *M N* = weight of the water *a b A B*. But *a b A B* is the increase of volume due to the immersion of *Q N*, and is therefore equal in bulk to *Q N*. Hence the weight of the body *M N* is equal to the weight of a mass of water having the same bulk as the immersed part of the solid.

PRACTICAL ILLUSTRATIONS OF THE FORCE OF BUOYANCY.

684. In raising a bucket in a well, we find that a very slight force suffices to cause its ascent through the water; but as soon as it begins to emerge, its weight seems to increase, and this effect augments until the bucket is raised entirely out of the water. While it is submerged, the bucket is supported by a buoyant force equal to the weight of the displaced volume of liquid, and this, but for the weight of the iron handle and loose chain, would of itself raise the bucket to the surface. As the bucket emerges, the buoyant force diminishes and the chain is called upon to support more and more of the weight, until, when the bucket is entirely out of the water, its whole weight is transferred to the chain.

685. Heavy masses of rock or other materials, when surrounded with water, are capable of being moved much more easily than when on dry land. For the buoyant force diminishes their pressure upon the bottom, and to this extent diminishes the friction which opposes our efforts to slide them on the surface, and by sustaining a part of their weight aids us in lifting them or rolling them over. A mass of limestone one cubic foot in bulk weighs 2720 ounces or 170 pounds; but surrounded by water its pressure downwards is reduced to $170 - 62\frac{1}{2} = 107\frac{1}{2}$ pounds, or less than two-thirds of its actual weight. Hence it can be moved forward on the bottom, or lifted or rolled over by a force about two-thirds of that which would be necessary if the water were not present.

This buoyant action of the water aids the moving force of rivers and tides, in conveying the particles of sand and clay which they bear along with them, in their customary flow; and it contributes to their transporting energy when, in their more violent movements, they sweep before them the large pebbles and even huge masses of rock that lie along their course.

686. In the floating and swimming of animals, we see many curious illustrations of the effect of the buoyant force or upward pressure of the liquid.

Aquatic birds, as ducks and geese, are so light compared with water, as to float upon its surface with but a small part of their bodies immersed. Many insects are able to walk nimbly on the surface of water, while only the extremities of their long legs are depressed below the level of the liquid. In this case, close observation will show that around each foot the water is made to recede on all sides, so as to form a cavity greatly larger than the foot. This apparent repulsion of the water belongs to the class of molecular actions, treated of under the head of Capillary Force, and cannot be explained in this place. The amount of water thus displaced by the feet of the insect is such as to equal in weight the whole body of the animal, and thus the buoyant force is able to sustain this weight without further immersion.

The weight of the human body, during life, is in most cases a very little greater than that of an equal bulk of river water, and is generally a little less than the same volume of sea water. Hence there are probably few persons who would not float at the surface of the sea in calm weather. Corpulent people are, bulk for bulk, lighter than those of sparer habits, for the fat of animals is lighter than water, while lean flesh and bone are heavier. But for the cavities of the body, especially the lungs, the human frame would be much heavier than water. In the act of inspiration these are so much enlarged that, for the time, the body floats without requiring any effort to support it; but in the subsequent expiration the volume of the chest is contracted, and the buoyant force being thus diminished, the body sinks unless prevented by the efforts of the swimmer.

It is obvious from this, that a very small addition of buoyant force is sufficient to enable a person to float securely in water which is not agitated, and hence a single oar has often sufficed to save the lives of many who have together shared its support. In the act of swimming, the hands and feet are so moved as to give rise to a reaction of the water in an upward and a forward direction. The upward force, however, is greatly less than the other. All that is required is a force sufficient to keep the head above water; more than this would seem to be a hurtful expenditure of force when the object of the swimmer is to move forward.

The various contrivances for sustaining the body in water, known as swimming girdles, life preservers, &c., consist in the main of one or more pouches of gum elastic cloth, or other impervious material, which when needed can be inflated by the mouth. They all act upon the simple principle of increasing the immersed volume without much additional weight, and of thereby increasing the buoyant force.

687. A ship floating in quiet water displaces a weight of the liquid equal to the united weights of the hull, masts, sails, rigging, cargo, crew, and every thing else on board. It is obvious that, when loaded, her immersion is greater, or, in nautical language, she draws more water than when empty. Since a cubic foot of fresh water weighs 62½ pounds, it follows that, for each addition of 62½ pounds to the weight of the vessel, her immersion must be increased one cubic foot, and for each abstraction of the same quantity her immersion must be lessened one cubic foot. Hence, when by mischance a vessel has been run aground, the usual, and in most cases the only, mode of relief is to diminish her weight by casting overboard her guns and other heavy articles, thus lessening her immersion and enabling her to float off from the shoal or bar.

One of the large Atlantic steamers takes out on each voyage about 600 tons of coal. As this is nearly all consumed on the way, the weight of the great floating mass is diminished to this extent, and of course the depth or her immersion is correspondingly lessened. Hence the paddles, which at the beginning of the voyage plunge deeper into the water than is advantageous, are gradually lifted out at the same time that the hull of the ship rises so as to expose less surface to the resisting action of the water. In consequence of these changes, it is found that the speed of the vessel is greater during the latter than during the earlier half of her voyage.

688. Supposing the exact form and dimensions of a vessel to be known, we have it in our power to compute the volume of water which she displaces at each degree of immersion, and in this way it is practicable to weigh the ship and her cargo. Let *A B* (fig. 387) be the line to which the vessel sinks when unloaded, and let *C D* be the line which marks her immersion after she has received her cargo. The weight of the vessel without load is equal to the weight of the volume of water *A B N*. The weight of the vessel and

load united is equal to that of the volume of water *C D N*. Therefore the weight of the cargo by itself is the difference of these, that is, the weight of the volume of water *A B C D*. To find this, we would compute the number of cubic feet in the volumes *A B N* and *C D N* respectively, take their difference, and multiplying this by 62½, the result would be the weight of the cargo in pounds.

689. It often happens that the entrance to a river or harbor is obstructed by shoals, which are impassable by large vessels with their entire cargoes on board. In such cases the expedient commonly used is to transfer part of the cargo to large flat-bottomed boats called lighters, by which means the immersion of the vessel is so much reduced as to allow her to pass freely over these obstructions. But the same result can be attained without the troublesome process of removing a part of the load. This is effected by means of what is called the *Water Camel*, an ingenious contrivance which seems to have been used in China from a very early period, and which has been long employed to aid the navigation of the shallow rivers emptying into the Zuyder Zee, in Holland.

The water camel (fig. 388) consists of two huge wooden boxes or reservoirs made water-tight, and furnished each with valves beneath, for the admission of water, and with powerful pumps by which it is afterwards withdrawn. These are brought along-side of the ship, one on each side, and by opening the valves below are allowed to sink to a proper position. The valves are then closed, and the reservoirs are secured to the vessel by strong chains or ropes carried around and beneath her hull. At this stage of the operation the vessel floats at her usual depth. But the pumps are now brought into action, and as the water is removed the whole mass, including both the ship and the camels, rises gradually to a less immersion, until at length it is capable of floating freely in the shallow water through which it is to be conveyed.

It is obvious that, for every cubic foot of water discharged by the pumps, the floating mass is made lighter by 62½ pounds. Hence the weight of water which it displaces is diminished by an equal amount, and therefore the volume of displacement or the immersion is lessened one cubic foot for every cubic foot withdrawn from the camel. Supposing 40,000 cubic feet to be thus pumped out, the immersion would be diminished by an equal volume, or in weight by more than 1000 tons, so that the largest ship, with her entire cargo, would be enabled to float in comparatively shallow water.

690. On the same principle a large water-tight box or float (fig. 389) is sometimes used for recovering heavy bodies which have sunk in tide-water. The box filled with air is floated out into the neighborhood of the body when the tide is high, and when the water subsides the heavy mass is secured to it by a strong rope or chain. As the tide rises again, it lifts the float and with it the ponderous mass, which while thus suspended is readily conveyed to the shore.

If we suppose the float to be a rectangular box 12 feet long, 6 feet wide, and 6 feet deep, and that by its own weight it sinks to the depth of 1 foot, it will be easy to compute its entire lifting power when employed as above described. Imagining it to be entirely submerged, it would displace $12 \times 6 \times 6 = 432$ cubic feet of water, and the buoyant force of the liquid would be equal to the weight of this volume. But the weight of the box is equal to that of $12 \times 6 \times 1 = 72$ cubic feet. Hence the power which would be available in lifting or suspending the attached mass, is equal to the weight of $432 - 72$ or 360 cubic feet $= 360 \times 62.5$ pounds, or nearly 10 tons.

691. When it is required to maintain at a uniform level the water in a reservoir which is fed by a variable stream, an apparatus called the Ball-cock is used. This is a large floating ball of wood or thin metal (fig. 390), attached by a rod to the handle or lever of the cock, by which the water is supplied. The dimensions of the parts are such, that when the water rises above the standard line, its buoyant force acting on the ball turns the handle of the cock so as to contract or close the passage for the entering liquid, and when the surface descends below the standard level, the ball, now only partially supported by the water, exerts a downward force, which, being transmitted by the rod, turns the handle of the cock in the opposite direction, and thereby enlarges the passage through which the water is supplied.

692. The supply of water to the boiler of a steam-engine is in some cases regulated by a similar arrangement. Let *B* (fig. 391) represent the boiler, *M N* the pipe which supplies it with water, and *W* the float which is designed to regulate the supply. Suppose *H R* to be the level at which the water is to be kept. The rod, to which the float is fastened, is connected at top with one end of a lever *a b*, whose fulcrum is at *c*. The valve *V*, which controls the entrance of water through the supply pipe, is attached to the lever on the other side of the fulcrum. As the water is rapidly converted into steam which is conveyed to the engine, its level in the boiler sinks. As this diminishes the immersion of the float, and thereby lessens the buoyant force acting upon it, the float descends, and, in so doing, raises the valve. The water now flows into the boiler until the former level is restored, when, by its upward pressure on the float, it moves the lever the contrary way and thus closes the valve. As this regulating action commences the moment the level of the water begins to subside, the supply of liquid is made to correspond with the amount which is boiled off as steam, and thus the surface is maintained at a uniform level.

693. In Farey's self-acting flood-gate the buoyant force of the water is made to open the gate, whenever the surface of the liquid rises above a certain level in the pond or canal.

The gate *A B* (fig. 392) is moveable about a horizontal axis, of which one end is denoted by *a*. This axis is in a position *higher than the centre of pressure*, so that the resultant force of the water pressing upon the gate tends to keep it shut. An arm *a b* projects at right angles from the gate, and is connected by a joint with a vertical arm *b c*, to which is attached a large floating cask. A third arm, *D c*, moveable around *D* and *c*, prevents any inconvenient motion of the cask laterally. Let *M N* be the highest level at which the water is to be allowed to remain. The parts are so proportioned, that the buoyant force of the water at that level upon the cask is not sufficient by its action at *b* to turn the gate. But when the level rises above this, the buoyant action transmitted through *C b* revolves the gate into the position $A_1 B_1$ and allows the excess of water rapidly to escape. As the level subsides, the gate, in virtue of the preponderance of pressure below the axis *a*, is turned the opposite way until it is again closed.

694. Comparative immersion of a body in liquids of different densities.—It is obvious that the heavier the liquid in which a floating body is placed, the less will be its immersion, and the lighter the liquid the deeper it will sink. The mass *MN*, (fig. 393), which in water floats at the level *AB*, when transferred to the lighter liquid alcohol sinks to the level *CD*, and when placed in the far heavier liquid, mercury rests almost on the surface as at *EF*. Since in each of these cases the weight of the displaced liquid is equal to that of the floating body, and since this latter is the same in all the experiments, it follows that the weights of the volume of water *ANB*, the volume of alcohol *CND*, and the volume of mercury *ENF*, are equal to one another.

Assuming D to represent the weight of a cubic inch, or other unit volume of water, at a given place on the earth's surface, and d to represent that of an equal volume of alcohol at the same place, it is obvious that the density of the water will be to that of the alcohol as D to d. If now we estimate in cubic inches the volumes ANB and CND, which a given floating mass displaces in these two liquids respectively, we will have the weight of ANB of water $= D \times ANB$, and the weight of CND of alcohol $= d \times CND$. But these weights are equal, and therefore $D \times ANB = d \times CND$. Whence we have $ANB : CND = d : D$. That is, the displaced volume of water is to the displaced volume of alcohol, as the density of alcohol to the density of water. As a like result may be proved for all other liquids of unequal density, the following general proposition is true, viz.: *A floating body placed successively in liquids of different densities, will adjust itself so that the displaced volumes shall be inversely proportional to the densities of the liquids.*

695. When the floating body has the form of a prism or cylinder, as *MN* (fig. 394), the volumes ANB and CND are severally equal to $AH \times$ Area N, and $CH \times$ Area N. Therefore $ANB : CND = AH : CH$. But we have just seen that $ANB : CND = d : D$; whence it follows that AH:

$CH = d : D$. Here AH and CH are the depths of immersion at which the body is in equilibrium in the two liquids severally, and D and d are the densities of the liquids in which these immersions occur. Hence it follows, that *in the case of a prismatic or cylindrical mass floating vertically, the depths of immersion in different liquids are inversely proportional to the densities of the liquids.*

OF THE DIFFERENT KINDS OF EQUILIBRIUM OF A FLOATING BODY, AND THE CONDITIONS BY WHICH THEY ARE DETERMINED.

696. The equilibrium of a body floating in a liquid, varies with the position in which it rests, and like that of a mass suspended by a cord, or resting on a solid support, may be either *stable*, *tottering*, or *neutral.*

We have already seen, that, in whatever position the body may rest, the weight and the force of buoyancy must always be equal, and in the same vertical line. But although, in changing the attitude of flotation, there is no change produced in the volume of displacement, the height of the centre of gravity is in most cases either increased or diminished. Thus, when a rectangular prism floats with its longer side in a horizontal position as represented in (fig. 395), and is then turned around so as to have this side in a vertical direction, as in (fig. 396), its centre of gravity originally at the height GP from the surface of the liquid, rises to the greater height GP. It is evident that this ascent begins as soon as the mass is turned in the slightest degree from the position of (fig. 395), and that it continues until the mass is brought into the attitude of (fig. 396). Hence G is the lowest, and G_1 the highest attainable position of this point.

697. The floating body may rest permanently in either of the positions above represented. But any force which turns it from its position of rest will give rise to opposite results in the two cases. In the first attitude, (fig. 395), the disturbance of the mass will cause its centre of gravity to rise, but when the applied force is withdrawn, the body will oscillate until this point returns to G, its lowest attainable position. In the attitude represented in (fig. 396), the slightest disturb-

ance will cause the centre of gravity to descend. The body therefore, instead of recovering its position, will continue to turn over until the centre of gravity reaches its lowest place, when it will again be brought to rest. Thus it appears that, in the first of these attitudes of the floating mass, its equilibrium is stable, and in the second tottering.

In certain cases, as, for example, where the floating body is a homogeneous sphere, (fig. 397), it will rest indifferently in any position into which it may be turned, showing no tendency either to restore itself to its former attitude, or to continue its motion farther. In this case it is obvious, that the height of the centre of gravity is not changed by the motion of the mass; but this point either remains in the same place, or is moved in a horizontal direction.

698. The conditions which determine the kind of equilibrium in which a floating body rests are, therefore, as follows: —

When on turning the mass the centre of gravity is made to rise, the equilibrium is stable. When it is made to descend it is tottering, and when the motion of this point is neither upwards nor downwards, the equilibrium is neutral.

It will be remarked, that these are the conditions formerly demonstrated for the equilibrium of a body suspended by a cord, or sustained by a select support.

699. MOTION OF THE CENTRE OF BUOYANCY CAUSED BY THE ROLLING OF THE FLOATING MASS. When the body *A B C D*, (fig. 398), is turned from its erect position into the oblique attitude *a b c d*, (fig. 399) the centre of buoyancy shifts its place from *B*, the centre of gravity of *H C D R*, to *b* that of *h c d r*. The direction of this motion must evidently be towards the side where the displacement is increasing, and its amount must depend upon the rapidity of this increase compared with the entire displacement. As the disturbing force is supposed to act so as to produce rotation only, it may be assumed that the floating mass is neither depressed nor raised by it, and hence, in the oblique as well as the erect position, displaces a weight of liquid equal to its own weight. From this it follows, that the volume of the displacement is the same in all attitudes of the floating mass. Thus, in (fig. 399), *h c d r*, the volume displaced in the oblique position is equal to

$HcdR$, that which is displaced when the body is undisturbed. We may, therefore, consider $hcdr$ as formed by the transference of the triangular mass of liquid PHh to the position PRr, on the other side of the displaced volume.

700. To show the effect of this transference in shifting the centre of buoyancy B, let g be the place of the centre of gravity of the triangular mass before its transfer, and g_1 its place afterwards, and S be the centre of gravity of the mass $hcdRP$. Since B is the centre of gravity of $HcdR$, that is, of $hcdRP$, and the triangle PHh united, this point must lie in the line Sg, which connects their respective centres of gravity, and must divide this line in the inverse ratio of these masses. Hence, $BS : Bg = PHh : hcdRP$. Joining S and g, it is evident that the centre of gravity of $hcdr$, the oblique displacement, will be in the line Sg_1 at a point which divides it in the inverse ratio of $hcdRP$ and PBr. The point b thus placed is obviously the centre of buoyancy corresponding to the oblique immersion.

Since $bS : bg_1 = PRr : hHcdRP$, it follows by comparison with the preceding proportion, that $BS : Bg = bS : bg_1$ and therefore the lines Sg and Sg_1 are similarly divided at B and b. Hence Bb is parallel to gg_1. It thus appears that the motion of the centre of buoyancy, caused by turning the floating mass into an oblique position, is parallel to the motion of the centre of gravity of the transferred triangle.

To find the extent of this motion, in other words, the length of Bb, we proceed as follows: Since by what precedes $BS : Bg = PHh : hcdRP$, we have by composition $BS : BS + Bg = PHh : PHh + hcdRP$; that is, $BS : Sg = PHh : HcdR$. But by parallel lines
$BS : Sg = Bb : gg_1$. Whence we have
$Bb : gg_1 = PHh : HcdR$, which gives

$$Bb = gg_1 \times \frac{PHh}{HcdR}.$$

It follows, therefore, that, *by the turning of the floating mass, the centre of buoyancy is shifted in a line parallel to the path of the centre of gravity of the transferred prism,* *and through a distance which is equal to the length of this path multiplied by the volume of the prism, and divided by the whole displacement.*

OF THE METACENTRE AND ITS RELATIONS TO THE EQUILIBRIUM OF A FLOATING BODY.

701. We have seen that in all positions of a body floating in a liquid, the supporting force acts through the centre of buoyancy in a direction vertically upward, while the weight of the mass acts through its centre of gravity in an opposite direction, and that these forces are equal to one another. Let ANC, (fig. 400), represent a floating body which is deflected into an oblique position, its centre of gravity being at G, and centre of buoyancy at b. Let LGN be the axis of symmetry of the mass, or the line which, in the undisturbed position of the mass, divides it vertically into two equal and similar segments. It is evident that a vertical drawn upwards through b will intersect this line. Suppose M to be the point of intersection; then, by the transferability of forces, the force of buoyancy acting in the direction of the vertical bM, may be regarded as applied at the point M. By turning the floating body through a large angle, the position of the point M on the axis will be materially changed; but if we suppose the rolling motion to be confined to a very small rotation of the mass, on each side of its vertical position, the vertical lines drawn through the centre of buoyancy, in all its successive positions, may be regarded as meeting at M; and thus the force of buoyancy may be considered as continually directed towards this point. Under these conditions the point M is called the METACENTRE.

We may, therefore, define the metacentre to be that point in the axis of symmetry of a floating body in which all the directions of the force of buoyancy meet, when the angle of rotation of the body is very small.

702. Let MQ (fig. 400) represent the force of buoyancy, and let the equal line GW represent the weight of the floating mass. These two forces, instead of producing equilibrium, will obviously tend to rotate the mass. In the present case, the rotation will be towards the vertical line, and will there-

fore restore the body to its erect position. When, however, the centre of gravity is situated higher up on the axis than the metacentre M, as for example at G_1, the forces MQ and G_1W conspire to turn the mass away from the vertical, and thus finally to overthrow or capsize it. When, again, the centre of gravity G and the metacentre M coincide, the upward and downward forces are directly opposed, and therefore neutralize one another. Hence the floating mass neither returns to the vertical position, nor rolls over in the other direction, but retains the oblique attitude into which it has been deflected.

703. It appears, therefore, that the kind of equilibrium presented by a floating body may, in all cases, be determined by the relative position of the metacentre and centre of gravity, according to the following rules.

First. — *When the centre of gravity of a floating body is lower than the metacentre, the equilibrium is stable.*

Second. — *When the centre of gravity is higher than the metacentre, the equilibrium is tottering.*

Third. — *When these points coincide, the equilibrium is neutral.*

704. OF THE MEASURE OF THE STABILITY OF A SHIP OR OTHER FLOATING BODY.

Let ANC (fig. 401) represent a transverse section of the body, supposed to include both the centre of gravity and the centre of buoyancy; and let G and b denote the position of these points respectively. Drawing the vertical bM, and the line GV perpendicular to it, and taking the equal lines VP and GW to represent severally the forces of buoyancy and gravity acting on the mass, it is obvious that these two forces form a *statical couple*, whose arm is GV. The moment of this couple being the product of either of the equal forces into the arm of the couple, may be expressed by Weight × GV or Buoyant force × GV. This moment measures the effort of the couple to turn the floating body towards the vertical position, and is therefore a proper measure of the power of the body to resist a force tending to overturn it; in other words, it is a measure of the stability of the ship or other floating mass.

705. In order to compare the forces of stability of two

vessels, or of the same vessel when differently loaded, we assume them both to be deflected through a small but equal angle. Let ANC and anc (figs. 401, 402) represent two vessels so deflected from the vertical as to make the angles at M and m small and of equal magnitude. Drawing the corresponding lines in the two figures, and using ANC and anc to represent the weights of the two vessels respectively, we have by the preceding proposition:

Stab'y of ANC : Stab'y of $anc = ANC \times GV : anc \times gv$.

But since the triangles MGV and mgv are equi-angular, we have

$GV : gv = GM : gm$. Whence it follows that

Stab'y of ANC : Stab'y of $anc = ANC \times GM : anc \times gm$.

That is, *the stability is proportional to the product of the weight of the floating mass into the distance of its centre of gravity below the metacentre.*

706. It follows from this, that the stability of a ship is increased by increasing her weight, and by augmenting the distance of the centre of gravity below the metacentre. The conditions necessary for securing the greatest degree of stability in a ship are, first, that the centre of gravity of the mass shall be as near the keel as possible; and, secondly, that the form of the hull be such as to cause the centre of buoyancy to advance rapidly towards the descending side of the ship, so as to give the largest possible value to GV, and therefore to GM, with a given deflexion from the vertical.

The former of these conditions is obtained by stowing the cargo in the lowest position in which it can be placed, and by placing the heaviest articles nearest the keel. The other condition is best answered, by giving to the hull a very broad and swelling outline towards the plane of flotation. Thus, on comparing the sections shown in (figs. 401, 402), it is evident, that with the same inclination the volume of the transferred prism, PRr, is greater in the latter case than in the former, and at the same time the distance gg_1, through which it is moved, is greater. Hence the distance Bb, through which the centre of buoyancy advances, is also greater. This increases the line GV, and therefore raises the metacentre to a greater height GM above the centre of gravity. Hence, supposing the weights to be equal, and the distance of the centre of gravity from the plane of flotation to be the same in both cases, the stability will be greater when the section has the form of (fig. 402), than when it has the other shape.

On comparing the position of the metacentre, which corresponds to a *rolling* or transverse deflection, as in either of the preceding figures, with that in (fig. 403), where the movement is longitudinal, or what is called a *pitching* of the vessel, we see that in the latter case the metacentre is greatly higher than in the former, and hence the stability, or power to resist an overturning force, is much greater.

707. MODE OF FINDING THE CENTRE OF BUOYANCY AND CENTRE OF GRAVITY OF A SHIP. We have seen that the stability of a vessel depends upon the place of the centre of buoyancy of the vessel in her oblique attitude, compared with the position of the centre of gravity on the axis of symmetry. Hence it is necessary to find the place of both these points, in order to compute the measure of the stability.

708. To determine the *place of the centre of buoyancy*, we proceed as follows. Let $h\,N\,r$, (fig. 404), represent what would be the immersed volume in the erect attitude, and B the centre of buoyancy in this position. We know that B is the centre of gravity of the volume hNr. From the depth of immersion, and the known form of the hull, we have the means of determining the distance BN, or the place of the centre of buoyancy on the axis of symmetry, when the vessel floats in an erect position. Let now a small inclination be given to the deck by shifting a heavy weight W from D across the deck to C, or, for the sake of greater efficiency, to the end of a projecting spar at E. In consequence of the inclination thus caused, the centre of buoyancy will be shifted in the line Bb, parallel to gg_1, the path of the centre of gravity of the displaced prism.

To find the position of b, the new centre of buoyancy, we proceed as follows: Having measured the inclination or angle of deflexion X, we know its equal the angle RSr of the transferred prism, which, with the known form of the hull, enables us to compute the volume of this prism, as well as the path $g g_1$ of its centre of gravity, due to the displacement. With these data we determine the line $B b$, according to a previous proposition, (700), that is,

$$B b = \frac{\text{weight of prism} \times g g_1}{\text{weight of ship} + W}.$$

Since $B b$ is parallel to $g g_1$, it makes an angle with the horizontal line $B c$ equal to about one half the angle $R S r$, or half the inclination X, and as this is assumed to be very small, the angle at B will be so insignificant that we may regard $B c$ as equal to $B b$. Thus the place of the centre of buoyancy of the vessel, in her oblique position, is found by drawing from the known place B, of the centre of buoyancy for the erect attitude, a horizontal line $B c$ or $B b$, of a length given by the preceding equation.

709. To find the *position of the centre of gravity* G, that is, to determine its height, $G N$, above the keel, or its distance $G D$ from the deck, we proceed as follows: Through b draw the vertical $b M$, and through M draw a horizontal line to meet the verticals $G Q$ and $E P$, drawn from the centre of gravity of the ship, and of the weight W respectively. The whole force of buoyancy, equal to the weight of the ship plus W, acts upwards along the line $b M$, and may be considered as applied at M, while the weight of the ship may be regarded as acting at Q, and the weight W as acting at P. As these forces are in equilibrium, M may be viewed as a fulcrum or point of moments, and we have, weight of ship $\times M Q = W \times M P$.

$$\text{Whence } M Q \text{ or } G V = \frac{W \times M P}{\text{weight of ship}}.$$

In this expression W is supposed to be known. MP is computed from the distance $D E$, to which the weight has been shifted, and the inclination X of the deck, both of which are to be measured. It is obvious that MP is to DE in the ratio of XM to XD, that is, of unity to the cosine of the inclination, and therefore, $MP = \frac{D E}{\text{cos. inc.}}$. As regards the denominator of the above expression, viz. weight of ship, this is found by computing the whole volume of displacement in cubic feet, and then reducing to pounds, as formerly explained.

Lastly, from Bc and GV, as found above, we determine the place of G on the axis MN in the following way. Bc and GV being known, their difference Bd is known. The angle QGM is equal to $90^\circ - XMD$, and is therefore equal to X the inclination. Hence in the right-angled triangle GBd we have Bd, the angle at G, and the right angle to compute BG. Thus $BG : Bd = 1$: sin. inclination, and therefore, $BG = \frac{BD}{\text{sin. inc.}}$. Having previously determined the distance of B from the keel or deck, that is, BN or BD, we may now find the place of G in the axis, by the equation $GN = BN - BG$, or $GD = BD + BG$.

710. *Effect of a side wind on a vessel.* — Let ANC, (fig. 405), represent a vessel maintained in an oblique position by the action of the wind. It is proposed to show in what way the force of the wind, while it balances the action of gravity and the buoyant force, both of which tend to bring her back to a vertical position, impels the vessel transversely; that is, in the direction of the wind. Suppose S to be the point of the axis at which the resultant of the whole force of the wind is applied, and let SW denote its direction and intensity. Assuming G and B as the centres of gravity and buoyancy respectively, and drawing verticals through these points, take the equal lines GQ and MP to represent the directions and intensity of the force of gravity and that of buoyancy. Bisect GM in O, and prolong the axis downwards to S, a distance equal to OS. We may regard this as a rigid line connected with the vessel. Imagine two forces, W_1 and W_2, each equal to SW to be applied at S, in opposite directions. As they would balance one another, they would not in any way modify the effect produced by SW.

We may thus consider the three forces W, W_1 and W_2 as equivalent in their action to W. Hence the ship may be viewed as a mass simultaneously acted upon by the five forces GQ, MP, W, W_1 and W. Of these, the two first form a statical couple tending to rotate the ship back to the vertical position, the third and fourth, W and W_1, form a statical couple tending to roll the ship over in the opposite direction. But as, by our supposition, the inclination of the vessel remains steady, these opposing forces must be in equilibrium; that is, the moment of the couple W, W_1 must be equal to that of the couple $G\,Q$, $M\,P$. There remains then the fifth force W_2, which is equal to that of the wind, and which impels the ship in the transverse direction $S_1\,W_2$, or $S\,W$, causing the sidelong motion, called by sailors, the lee-way.

CHAPTER XXXI.

OF SPECIFIC GRAVITY, AND THE METHODS BY WHICH IT MAY BE DETERMINED.

711. The *absolute weight* of a body, that is, the entire force with which it tends to the earth, as measured by its action on the spring balance, is dependent on three conditions; first, the intensity of the force of gravity acting on each particle; secondly, the number of particles contained in each cubic unit of the mass; and, thirdly, the number of such units included in its whole bulk. If g denote the force of gravity acting on each particle, measured by the acceleration which it would impart in 1″, and D and V denote the density and volume of the body respectively, then the absolute weight W will be measured by $g \times D \times V$, or the product of gravity and the density and the bulk of the mass.

712. As the value of g is slightly varied by a change of latitude, and by difference of height above the earth's surface, it follows that the absolute weight of a given mass of matter is slightly changed by these changes of position. If, however, we compare the obsolute weights of two masses at the same place on the earth's surface, the value of g will be the same for both. Using the large letters for one, and the small letters for the other body, we will have

$W = g\,D\,V$ and $w = g\,d\,v$. Whence the proportion $W : w = D\,V : d\,v$.

That is, the absolute weights of bodies at a given place, are proportional to the products of the density and volume of each. Thus if the density of mercury be 13, and that of water 1, while the volume of the mercury is 3, that of the water being 12, their weights will be in the proportion of 13 × 3 to 1 × 12, or 39 to 12.

713. Let W and w denote the weights of equal bulks of any two bodies. Then, by the previous proportion, we have $W : w = D\,V : d\,v$. But since in this case V and v are equal, it follows that $W : w = D : d$. That is, at a given place, the absolute weights of equal bulks of any bodies are proportional to their densities.

714. The term SPECIFIC GRAVITY is used to denote the weights of equal bulks of different bodies, when compared with the weight of the same bulk of some standard substance. The substance used as the standard for the specific gravity of all solid and liquid bodies, is *water*. That referred to as the standard for the specific gravity of gases and vapors, is *atmospheric air*.

Let C, B and A (fig. 406) denote equal bulks of lead, pine wood and water; then we will have Specific Gravity of Lead $= \frac{\text{Weight } C}{\text{Weight } A}$, and Specific Gravity of Pine Wood $= \frac{\text{Weight } B}{\text{Weight } A}$.

Thus the specific gravity of a substance is the number of times the weight of a given volume of this substance contains the weight of an equal volume of water.

It is obvious that the specific gravity of the water, or other substance used as a standard, will be unity. For using the above expression for finding its value, we have

$$\text{Specific Gravity of Water} = \frac{\text{Weight } A}{\text{Weight } A} = 1.$$

715. Since in the above expressions for specific gravity, the weight A of the given bulk of water is constant, it follows that the specific gravity of the lead is to that of the wood in the same proportion as the weights of C and B, the equal bulks of lead and wood. But the weight C bears to the weight A the same ratio as the density of the lead to that of the wood, (713). Hence the specific gravities of the lead and wood are in the same proportion as their densities.

Therefore in general, denoting by D and d the densities, and by S and s the specific gravities of any two substances, we have

$$S : s = D : d.$$

The term density is often used as synonymous with specific gravity, and the numbers which express the latter are also regarded as representing the former.

716. Let C, B and A (fig. 407) denote the unequal volumes corresponding to equal weights of lead, pine wood, and water. Using S s and 1 to mark their respective specific gravities, and substituting these for the densities in the proportion above given (712), we have Weight of C: Weight of $A = S \times \text{Volume } C : 1 \times \text{Volume } A$. But, by hypothesis, Weight of $C =$ Weight of A. Hence

$$S \times \text{Volume } C = \text{Volume } A, \text{ and therefore}$$

$$S = \frac{\text{Volume } A}{\text{Volume } C}.$$

$$\text{That is, Specific Gravity of Lead} = \frac{\text{Volume } A}{\text{Volume } C}.$$

$$\text{And so likewise Specific Gravity of Pine Wood} = \frac{\text{Volume } A}{\text{Volume } B}.$$

Therefore, in general, *the specific gravity of a substance is the number of times its volume is contained in the volume of an equal weight of water.*

717. Uniting this and the preceding expression, we are led to two general methods for determining the specific gravities of bodies instrumentally.

The first, or *method of weights*, consists in ascertaining the weight of the body, and the weight of an equal bulk of water, and then dividing the former by the latter. Thus,

$$\text{Spec. Grav. of Body} = \frac{\text{weight of body}}{\text{weight of equal bulk of water}}.$$

The second, or *method of volumes*, consists in finding the volume of the body, and the volume of an equal weight of water, and then dividing the latter by the former. Thus,

$$\text{Spec. Grav. of Body} = \frac{\text{vol. of equal weight of water}}{\text{vol. of body}}.$$

718. In both these expressions, *water* is assumed to be the

standard, and its specific gravity is, therefore, reckoned as unity, while the specific gravities of solid and liquid substances are denoted by the numbers which result from the formulæ above indicated. But it is obvious, that similar formulæ are applicable, whatever standard may be employed. When *atmospheric air* is used for the purpose, as in finding the specific gravities of gases, we proceed exactly as above, only substituting atmospheric air instead of water. Thus, to find the specific gravity of a gas by the method of weights, we divide the weight of the gas by the weight of an equal bulk of atmospheric air; to find it by the method of volumes, we divide a given bulk of atmospheric air by the bulk of an equal weight of the gas.

719. Illustration of the above general method by examples.

1st. Let the weight of a given bulk of lead be 66 ounces, and that of an equal bulk of water be 5.8 ounces, then by the first method, we have

$$\text{Spec. Grav. of Lead} = \frac{66}{5.8} = 11.3.$$

2d. Let the weight of a given bulk of wood be 12 ounces, and that of an equal bulk of water be 32 ounces, we have, by the first method,

$$\text{Spec. Grav. of the Wood} = \frac{12}{32} = 0.375.$$

3d. Let the volume of a given weight of sulphuric acid be 25 cubic inches or other units of bulk, and that of an equal weight of water be 46 of the same units, then, by applying the second method, we have

$$\text{Spec. Grav. of Sulphuric Acid} = \frac{46}{25} = 1.84.$$

4th. Let the volume of a given weight of alcohol be 56 cubic units, and that of an equal weight of water be 35 of the same units, then, by the second method, we have

$$\text{Spec. Grav. of Alcohol} = \frac{35}{56} = 0.625.$$

5th. The weight of one cubic foot of atmospheric air, in standard conditions, is 554.8 grains; the weight of one cubic foot of chlorine gas, in like conditions, is 1370 grains. Hence, by the first method, we have

Spec. Grav. of Chlorine Gas $= \frac{1370}{554.8} = 2.48$, the specific gravity of air being unity.

720. Of the standard temperature for specific gravities. —It is a well known fact, true of all bodies, that a change of temperature is accompanied by a change of volume, the general law of this change being that the bulk of a body augments as its temperature is raised. Were the rate of this expansion the same for all substances, it would follow that, taking equal weights of any number of bodies at a given temperature, the corresponding volumes would continue at all temperatures to bear the same proportion to one another, and that therefore the specific gravities of the bodies would remain unaltered, at whatever common temperature they may first have been determined.

This hypothesis, however, is not correct. Each solid and each liquid has a rate of expansion proper to itself, and even the gases differ slightly in their expansibility, although the difference is so small as to be practically insignificant. It follows, from this, that the specific gravities of the same bodies, whether determined by the method of weights or of volumes, will vary with the temperature at which they are instrumentally ascertained. In illustration of this result, suppose L and W (fig. 408) to represent equal bulks of lead and of water, both at the temperature 32°. Let L_1 and W_1 be the volumes of the same masses when raised to the temperature of 100°. The volume L_1 consists of two parts, viz., l which is equal to the original bulk L, and the addition caused by the expansion. This addition is some particular fraction of l, which may be expressed by $m\ l$, in which m represents the ratio of the expansion of the lead to the original bulk. So likewise the volume W_1 consists of two parts, viz. w, which is equal to the original bulk W and the addition caused by the expansion. This addition may be expressed by $n\ w$, in which n denotes the ratio of the expansion of the water to its original bulk. Now by the method of weights, we have specific gravity of

Lead at 32° $= \frac{\text{weight of } L}{\text{weight of } W}$.

But weight of L = weight of $(l + m\,l) = (1 + m)$ weight of l. In like manner weight of $W = (1 + n)$ weight of w.

$$\text{Whence S. G. of Lead at } 32° = \frac{(1+m)\text{ weight of } l}{(1+n)\text{ weight of } w}.$$

$$\text{But S. G. of Lead at } 100° = \frac{\text{weight of } l}{\text{weight of } w}.$$

Since water is more expansible than lead, n is greater than m, and therefore $1 + n$ is greater than $1 + m$. Whence the first of the above expressions is less than the second; in other words, the specific gravity of lead at 32° is less than at 100°.

721. In some of the tables of the specific gravity of solid and liquid bodies, the standard adopted is the temperature of melting ice, that is, 32° Fahrenheit. In others the temperature of 39.2°, and in others again those of 60° and 62° are used. The first of these has the advantage of being marked by an obvious natural change, the melting of ice, which always occurs at exactly the same temperature. The water, in which ice is melting, soon acquires this temperature, and retains it without variation as long as any considerable amount of the ice remains unmelted. It is in our power, therefore, to secure this temperature exactly, without relying on the imperfect and variable accuracy of the thermometer. The second standard temperature, 39.2°, is that at which pure water has its greatest density. A cubic foot or other given volume of the liquid, measured at this temperature, contains more matter, and therefore weighs more than the same volume taken at any higher or lower temperature. Starting from this point of the scale, experiment shows that, whether we heat or cool the water, its bulk is increased. There does not, however, appear to be any special advantage resulting from the choice of this standard, and, in practice, it is a temperature not easily secured and kept up.

The temperatures of 60° and 62° are more convenient than either of the preceding in the practical determination of specific gravities; for as they do not differ greatly from the prevailing temperature of the air and water in middle latitudes, the operator finds no difficulty in bringing the substance on which he is experimenting as well as the standard water to either of these temperatures. In common observations for determining the specific gravity of minerals, mineral waters, and many other solid and liquid substances, it is the practice to reduce them to 60° Fahr. before weighing or measuring them.

The specific gravity of gases and vapors are usually determined at 60° or 62°; but in the best tables they are referred to 32°, the numbers corresponding to the latter temperature being computed from the experimental results at 60°, by allowing for the expansion due to the increase of temperature. In determining the specific gravity of gases and vapors, however, it is not enough that the temperature be the same for all. It is also necessary that the pressure acting upon them be the same, for owing to their ready compressibility, a small addition to the force by which they are compressed is sufficient to cause a marked increase in their density. The standard pressure to which the specific gravities of gases and vapors are referred, is that of a column of mercury 30 inches high, which is very nearly the average pressure of the atmosphere at the level of the sea.

The most recent tables of the specific gravities of solids and liquids adopt 39.2° as the standard temperature, but in most cases the specific gravities so recorded have not been directly determined at this temperature, but have been computed from the results of observations, made at the more convenient temperature of 60°.

722. The method of deducing the specific gravity corresponding to any given temperature from the known specific gravity at another temperature, will readily appear by comparing the formulæ before given (720) for the specific gravity of a mass of lead at 32° and 100°. As there is nothing in these expressions limiting them to a particular substance or particular temperatures, we may apply them in a more general sense.

Let S be the specific gravity of a body as actually determined at the temperature T, and let s be the specific gravity of the same substance at any other temperature t. Also let m denote the expansion which the body suffers in being heated from T to t, and n that which water sustains by the same rise of temperature, the quantities m and n being fractions of the entire bulk of the body and the water at T.

Knowing the rates at which the water and the body severally expand for each degree, it is easy to complete the values of m and n. Now by the formulæ (720), we have

$$\text{Spec. Grav. of body at } T = \frac{(1+m) \text{ weight of body}}{(1+n) \text{ weight of water}},$$

$$\text{and Spec. Grav. of body at } t = \frac{\text{weight of body}}{\text{weight of water}}.$$

$$\text{Whence we have } S = \frac{1+m}{1+n} \cdot s.$$

$$\text{and therefore} \quad s = \frac{1+n}{1+m} \cdot S.$$

This formula may be expressed as follows. To find the specific gravity of a body at any given temperature from the known specific gravity at some given lower temperature, we divide the sum of unity and the expansion of the body due to the difference of temperature, by the sum of unity and the corresponding expansion of water, and then multiply the quotient by the specific gravity at the lower temperature.

When it is proposed to deduce the specific gravity at a lower from that at a higher temperature, as for example, from 60° to 32°, the quantities n and m must be made subtractive. For in this case the body and the standard substance are both of them contracted by the reduction of temperature, a cubic inch or other unit volume of the body being reduced to $1 - m$, and the same volume of water to $1 - n$. Hence in such cases the formula becomes

$$s = \frac{1-n}{1-m} \cdot S.$$

723. *To compute the absolute weights and volumes of bodies, whose specific gravities are known.*— For the solution of these problems, we require to know the absolute weight of some known bulk of the standard substance. According to the most accurate observations, the weight of one cubic inch of water, at the temperature 62°, is 252.458 grains. The weight of a cubic foot, at the same temperature, is 997.125 ounces; and at 39.2, the point of maximum density, the weight of a cubic foot of water is 998.125 ounces. This number approaches so near to 1000, that for

ordinary calculations we may assume a cubic foot of water, at common temperatures, as equal to 1000 ounces, or 62½ pounds. A cubic foot of dry atmospheric air, at the temperature 32°, and under a pressure of 30 inches, has been found to weigh 1.291 ounces, or 554.8 grains.

724. In order *to compute the absolute weight* of a known volume of any liquid or solid, we have merely to multiply its specific gravity by 1000, and this product by its bulk in cubic feet. Thus, let it be required to find the weight of a mass of cast-iron, whose bulk is 12.5 cubic feet, the specific gravity of the metal being 7.2. Since 1 cubic foot of water weighs 1000 ounces, 1 cubic foot of the cast-iron will weigh $7.2 \times 1000 = 7200$ ounces, and therefore the whole mass or 12.5 cubic feet must weigh $12.5 \times 7.5 \times 1000 = 90000$ ounces.

725. *To compute the bulk of a mass* of known weight and specific gravity, we first find the weight of a cubic foot of the body by multiplying its specific gravity by 1000, and then we divide the whole weight of the mass by this. Thus, suppose a mass of anthracite coal, whose specific gravity is 1.8, to weigh 78000 ounces, then 1 cubic foot of the coal weighs 18000 ounces. Therefore the volume of the whole mass is $\frac{78000}{18000}$ cubic feet, or $4\frac{1}{3}$ cubic feet.

INSTRUMENTS FOR DETERMINING SPECIFIC GRAVITIES.

These may be divided into two classes, viz., those by which we compare the weights of equal bulks, and those by which we compare the bulks of equal weights.

First. — *Instruments for comparing the weights of equal bulks of the body and the standard.*

(*a*). THE HYDROSTATIC BALANCE, (fig. 409). — This is a delicate pair of scales, furnished with means of suspending a solid body beneath one of the scale-pans, so as to counterpoise it successively in the air and in water, or any other liquid.

726. To find the specific gravity of a *Solid* by this apparatus, we first weigh it in air, then bringing beneath it a vessel of water, we counterpoise it while immersed in the

liquid. Let A denote the absolute weight, and W the water weight of the body. Their difference, $A - W$, is evidently due to the supporting force of the water, and this by the principle of Archimedes is equal to the weight of the volume of water displaced by the body. Thus we have

$A =$ the weight of the body.
$A - W =$ the weight of an equal bulk of water.

Hence the specific gravity of the Body $= \frac{A}{A - W}$. When using this method, therefore, the rule for computing the specific gravity is as follows. Having found the absolute weight and the water weight of the body, divide the absolute weight by the difference between it and the water weight, and the quotient will be the specific gravity.

727. If the solid be so light as not to sink in the water, we attach below it, to the same suspending line, a dense body capable of holding it immersed. Thus, let C (fig. 410) represent a fragment of cork, whose specific gravity is to be found, and L a mass of lead attached to the same suspending line. Let L denote the absolute weight, and l the water weight of the lead. These are to be determined by a preliminary experiment. We thence have $L - l$, the weight of the volume of water which is displaced by the lead. We now weigh the compound mass L and C in the water, and substract the water weight thus found from the sum of the absolute weights of L and C, determined by ordinary weighing. The difference by the principle of Archimedes, is equal to the weight of the volume of water displaced by both L and C. Taking from this the weight of the volume displaced by L as previously found, the remainder is the weight of the volume displaced by C. Dividing the absolute weight of the cork by the weight of an equal bulk of water thus found, we have the specific gravity of the cork.

728. In determining the specific gravity of *Liquids* by the hydrostatic balance, we proceed as follows. Suspend from the scale some solid body, such as a rounded mass of glass, upon which the liquids do not act chemically. First, determine its absolute weight A, then counterpoise it while plunged in water, and thus find its water weight W. The difference $A - W$ will be the weight of the displaced

water. Next, counterpoise it while immersed in the other liquid whose specific gravity is required. Let L denote its weight while plunged in this liquid. Then, by the principle of Archimedes, $A - L$ will be the weight of the volume of this liquid which it displaces. Thus we have $A - W =$ weight of the given bulk of water, and $A - L =$ weight of the same bulk of other liquid. Hence specific gravity of liquid $= \frac{A-L}{A-W}$. The rule, therefore, for determining the specific gravity of a liquid by the hydrostatic balance is as follows: Balance the suspended solid in air, then in water, and then in the given liquid, and find how much weight it loses by immersion, in the water and in this liquid respectively; divide the loss of weight in the liquid by the loss of weight in the water, and the quotient will be the specific gravity required.

(*b*). THE CONSTANT OR GUAGE FLASK. — This is a small glass flask, (fig. 411) with a slender neck, to which is fitted a glass stopper, perforated lengthwise by a small opening. It is of such size that, when filled with pure water at the temperature 60°, as high as the outer end of the perforation, the contained liquid weighs exactly 1000 grains. It is chiefly used for determining the specific gravity of liquids. For this purpose the liquid is brought to the standard temperature, and the flask at the same temperature is filled with it up to the outer end of the stopper. The whole is then accurately weighed, and the known weight of the flask and stopper being deducted, the remainder is the weight of the given bulk of liquid. We have, therefore, specific gravity of liquid $= \frac{\text{weight of contents of flask}}{1000}$.

Suppose that the flask being filled with common nitric acid, its contents weigh 1220 grains, then, specific gravity of nitric acid $= \frac{1220}{1000} = 1.220$. It is obvious that the specific gravity is found at once by shifting the decimal point as it stands in the weight of the contents of the flask, three digits to the left.

729. To use the guage flask for finding the specific gravity of a solid, we proceed as follows. We first find the

absolute weight of the solid. We then transfer it to the flask, and fill up the latter with water at the standard temperature. We now weigh the whole, and subtracting the weight of the flask, the remainder is equal to the joint weight of the solid and the water contained in the flask. Let B denote the absolute weight of the body, and w that of the equal volume of water which it has excluded from the flask. The weight of the water contained in the flask is obviously $1000 - w$, and that of the whole contents of the flask is $1000 + B - w$. Subtracting 1000 grains from the weight of the contents of the flask, we have the gain of weight $B - w$, which is due to the introduction of the solid. This quantity deducted from B, the absolute weight of the solid, gives w the weight of an equal bulk of water. Whence we have specific gravity of the solid $= \frac{B}{w}$. We have for this process, therefore, the following rule. From the weight in grains of the mixed contents of the flask subtract 1000, and take this remainder from the absolute weight of the solid; the result will be the weight of an equal bulk of water. Divide the absolute weight by this, and the quotient will be the specific gravity of the solid.

(*c*). NICHOLSON'S HYDROMETER. — This instrument (fig. 412), consists of a hollow vessel having conical ends, from the upper of which projects a wire surmounted by a small concave dish. From the lower end, by means of a hook, is suspended a heavy cone, whose upper surface forms a second dish, similar in shape to that above. This instrument is so constructed, that when loaded with a certain number of grains placed in the upper dish P, it sinks in pure water at the temperature 60°, to a fixed mark x on the wire. This is the standard mark, to which it must be adjusted when used for finding the specific gravity of solid or of liquid bodies.

730. To find the specific gravity of a *Solid* by Nicholson's Hydrometer.

Having placed the instrument in water of the standard temperature, remove the normal load from the upper dish. Place the solid in this dish, and then continue to replace the weights until the instrument sinks to the standard point x. The weight that remains off the dish is evidently equal to the absolute weight of the solid. Let this be denoted by A. Next, transfer the solid from the upper to the lower dish. The instrument will be seen to rise, because by the immersion of the solid in the water a part of its weight is supported by the buoyant force of the liquid. Add weights in the upper dish until the standard point x is again brought down to the liquid. The amount of weight thus added is evidently just equal to the buoyant force which the liquid exerts upon the immersed solid, and, by the principle of Archimedes, is equal to the weight of the water which the solid displaces. Let this weight be denoted by w. Then we have

A = absolute weight of solid,
and w = weight of equal bulk of water.

Whence specific gravity of solid $= \frac{A}{w}$.

731. To find the specific gravity of a *Liquid* by Nicholson's Hydrometer.

Let W be the weight of the hydrometer, and G that of the load or adjusting weight in the upper dish when the instrument sinks to the standard point x in water at 60°. Having brought the given liquid and the hydrometer floating in it to this temperature, we replace the weights above until the stem is brought to the standard point of immersion. Let the weights in the upper dish necessary to this result be denoted by g. In this condition the aggregate weight of the floating mass is $W + g$. When adjusted to the same depth in water, the aggregate weight is $W + G$. But we have seen (682) that the weight of a floating body is always equal to the liquid which it displaces. Therefore $W + g$ and $W + G$ are respectively equal to the weight of the displaced volume of liquid and the same volume of water. Hence we have; Spec. Grav. of the Liquid $= \frac{W + g}{W + G}$.

732. Second.—*Instruments for comparing the bulk of equal weights.*

These are used exclusively for finding the specific gravity of liquids. They consist of a hollow ball of metal or glass, (fig. 413), carrying beneath it a vertical stem loaded at the end, for the purpose of keeping the instrument in an upright position. Attached to the ball above is a cylindrical rod or tube, carrying a graduated scale for marking accurately the depths at which the instrument floats in different liquids. In those instruments which are designed for measuring the specific gravity of liquids lighter than water, the zero of the scale is placed at *m* near the bulb. In those intended for liquids heavier than water, the zero is placed at *n* near the top of the stem. (Figs. 413, 414).